Fuzzy Description Logics

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- Introduction
- Description Logics
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- Fuzzy Description Logics with Hedges

“Logic is everywhere . . .”
Knowledge Representation and Reasoning

- Knowledge representation and reasoning is at the heart of any intelligent system.
- Late 1960s and early 1970s:
  - Frames and semantic networks.
  - “Intuitive” semantics.
  - “Intuitive” algorithms.
  - No formal semantics.
  - Mistakes!
- Today:
  - Description logics.
  - Formal semantics.
  - Formal reasoning.
  - Formally derived algorithms.
  - No mistakes.
  - Algorithms are not slower!
Introduction

► Given an application domain (world).

1 Define relevant concepts of the domain, i.e., the terminology.
2 Specify properties of objects and individuals occurring in the domain, i.e., the world description.
3 Use reasoning as central service to infer implicitly represented knowledge.
   • Classification of concepts and individuals.
   • Computation of a sumsumption hierarchy.
4 Restrict expressiveness to obtain decision procedures.
5 Investigate computational complexity.

A Knowledge Representation and Reasoning System

- Description Language
- TBox
- ABox
- Reasoning
- KB

Application Programs

Rules
A Description Language – Syntax

- Restricted first order language.
  - Constants denoting individuals.
  - Unary relation symbols denoting concepts.
  - Binary relation symbols denoting roles.
  - Variables $X, Y, \ldots$.
  - Connectives $\neg, \wedge, \vee, \leftrightarrow, \rightarrow$.
  - Quantifiers $\forall, \exists$.
  - Punctuation symbols.
Concepts

- **Terms**: constants and variables are terms.
- **Atomic concepts**: $p(X)$, where $p/1$ is a unary relation symbol.
- **Atomic roles**: $r(X, Y)$, where $r/2$ is a binary relation symbol.
- **Concepts**:
  - Atomic concepts are concepts.
  - If $E(X)$ and $F(X)$ are concepts, then $\neg E(X)$, $(E(X) \land F(X))$, $(E(X) \lor F(X))$ are concepts.
  - If $R(X, Y)$ is a primitive role and $E(Y)$ a concept, then $(\exists Y)(R(X, Y) \land E(Y))$ and $(\forall Y)(R(X, Y) \rightarrow E(Y))$ are concepts.
- **Notation**
  - $A$ denotes an atomic concept.
  - $E, F$ denote concepts.
  - $R$ denotes a primitive role.
- **We sometimes omit $X$ and write $E$ instead of $E(X)$**.
Terminological Axioms

- **Terminological axioms are:**
  - **Definitions** \((\forall X)(A(X) \leftrightarrow E(X))\).
  - **Specializations** \((\forall X)(A(X) \leftarrow E(X))\).

- A finite set of terminological axioms is called **terminology** or **TBox** if each concept occurs at most once at the left-hand side of an axiom.

- In this lecture we consider only acyclic terminologies.
An Example Terminology

\( \forall X \)(woman(X) \iff (person(X) \land \text{female}(X))) \\
\( \forall X \)(man(X) \iff (person(X) \land \neg \text{woman}(X))) \\
\( \forall X \)(mother(X) \iff (\text{woman}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{person}(Y)))) \\
\( \forall X \)(father(X) \iff (\text{man}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{person}(Y)))) \\
\( \forall X \)(parent(X) \iff (\text{father}(X) \lor \text{mother}(X))) \\
\( \forall X \)(gndmother(X) \iff (\text{mother}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{parent}(Y)))) \\
\( \forall X \)(motherWoD(X) \iff (\text{mother}(X) \land (\forall Y)(\text{hasChild}(X, Y) \rightarrow \neg \text{woman}(Y)))) \)
Semantics – Concepts

Let $I = (\mathcal{D}, \cdot^I)$ be an interpretation.

- $[A]^I \subseteq \mathcal{D}$.
- $[R]^I \subseteq \mathcal{D} \times \mathcal{D}$.
- $[-E]^I = \mathcal{D} \setminus [E]^I$.
- $[(E \land F)]^I = [E]^I \cap [F]^I$.
- $[(E \lor F)]^I = [E]^I \cup [F]^I$.
- $[(\exists Y)(R(X, Y) \land E(Y))]^I$
  $\{ d \in \mathcal{D} | \text{there ex. } d' \in \mathcal{D} \text{ such that } (d, d') \in [R]^I \text{ and } d' \in [E]^I \}$.
- $[(\forall Y)(R(X, Y) \rightarrow E(Y))]^I$
  $\{ d \in \mathcal{D} | \text{for all } d' \in \mathcal{D} \text{ if } (d, d') \in [R]^I \text{ then } d' \in [E]^I \}$.

Abbreviation

$(E(X) \land F(X)) \quad \Leftrightarrow \quad E \sqcap F$
$(E(X) \lor F(X)) \quad \Leftrightarrow \quad E \sqcup F$
$(\exists Y)(R(X, Y) \land E(Y)) \quad \Leftrightarrow \quad \exists R.E$
$(\forall Y)(R(X, Y) \rightarrow E(Y)) \quad \Leftrightarrow \quad \forall R.E$
$(p(X) \lor \neg p(X)) \quad \Leftrightarrow \quad \text{top}$ with $[\text{top}]^I = \mathcal{D}$
$(p(X) \land \neg p(X)) \quad \Leftrightarrow \quad \text{bot}$ with $[\text{bot}]^I = \emptyset$
Semantics – Terminological Axioms

Let \( I = (\mathcal{D}, \cdot^I) \) be an interpretation.

- \( I \models (\forall X)(A(X) \leftrightarrow E(X)) \) iff \([A]^I = [E]^I\).
- \( I \models (\forall X)(A(X) \leftarrow E(X)) \) iff \([A]^I \subseteq [E]^I\).

Abbreviation

\((\forall X)(A(X) \leftrightarrow E(X)) \leadsto A = E\)
\((\forall X)(A(X) \leftarrow E(X)) \leadsto A \subseteq E\)
The Example Terminology – Abbreviations

\[(\forall X)(\text{woman}(X) \leftrightarrow (\text{person}(X) \land \text{female}(X)))\]
\[(\forall X)(\text{man}(X) \leftrightarrow (\text{person}(X) \land \neg \text{woman}(X)))\]
\[(\forall X)(\text{mother}(X) \leftrightarrow (\text{woman}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{person}(Y))))\]
\[(\forall X)(\text{father}(X) \leftrightarrow (\text{man}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{person}(Y))))\]
\[(\forall X)(\text{parent}(X) \leftrightarrow (\text{father}(X) \lor \text{mother}(X)))\]
\[(\forall X)(\text{gndmother}(X) \leftrightarrow (\text{mother}(X) \land (\exists Y)(\text{hasChild}(X, Y) \land \text{parent}(Y))))\]
\[(\forall X)(\text{motherWoD}(X) \leftrightarrow (\text{mother}(X) \land (\forall Y)(\text{hasChild}(X, Y) \rightarrow \neg \text{woman}(Y))))\]

\[\text{woman} = \text{person} \cap \text{female}\]
\[\text{man} = \text{person} \cap \neg \text{woman}\]
\[\text{mother} = \text{woman} \cap \exists \text{hasChild} \cdot \text{person}\]
\[\text{father} = \text{man} \cap \exists \text{hasChild} \cdot \text{person}\]
\[\text{parent} = \text{father} \cup \text{mother}\]
\[\text{gndmother} = \text{mother} \cap \exists \text{hasChild} \cdot \text{parent}\]
\[\text{motherWoD} = \text{mother} \cap \forall \text{hasChild} \cdot \neg \text{woman}\]
**ALC**

- **ALC** attributive language with complements.
- **Concepts** $E, F \rightarrow \text{top} | \text{bot} | A | \neg E | E \sqcup F | E \sqcap F | \exists R.E | \forall R.E$. 
- **Terminological axioms** $A = E | A \sqsubseteq D$. 

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Expansion of a Terminology

▶ Let \( \mathcal{T} \) be a terminology.
▶ An atomic concept \( A \) is said to be **defined** in \( \mathcal{T} \) if contains an axiom where \( A \) occurs in the left-hand side.
▶ An atomic concept is said to be **primitive** in \( \mathcal{T} \) if it is not defined in \( \mathcal{T} \).
▶ Expansion of \( \mathcal{T} \):
  ▶ While the right-hand side of an axiom in \( \mathcal{T} \) contains a defined concept.
    ○ Replace the defined concept by its definition.
  ▶ The resulting terminology \( \mathcal{T}' \) is called **expansion** of \( \mathcal{T} \).
▶ Properties
  ▶ Acyclic \( \mathcal{T} \) have a unique expansion \( \mathcal{T}' \).
  ▶ \( \mathcal{T}' \) may be exponential in the size of \( \mathcal{T} \).
  ▶ \( \mathcal{T} \) and \( \mathcal{T}' \) have the same primitive and defined concept.
  ▶ \( \mathcal{T} \equiv \mathcal{T}' \).
The Expansion of the Example Terminology

\[
\text{woman} = \text{person} \sqcap \text{female} \\
\text{man} = \text{person} \sqcap \neg (\text{person} \sqcap \text{female}) \\
\text{mother} = (\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild}. \text{person} \\
\text{father} = (\text{person} \sqcap \neg (\text{person} \sqcap \text{female})) \sqcap \exists \text{hasChild}. \text{person} \\
\text{parent} = (\text{person} \sqcap \neg (\text{person} \sqcap \text{female})) \sqcap \exists \text{hasChild}. \text{person} \\
\quad \sqcup ((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild}. \text{person}) \\
\text{gndmother} = (\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild}. \text{person} \\
\quad \sqcap \exists \text{hasChild}. ((\text{person} \sqcap \neg (\text{person} \sqcap \text{female})) \sqcap \exists \text{hasChild}. \text{person}) \\
\quad \quad \sqcup ((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild}. \text{person})) \\
\text{motherWoD} = ((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild}. \text{person}) \\
\quad \sqcap \forall \text{hasChild}. \neg (\text{person} \sqcap \text{female})
\]
Normalization of a Terminology

Let $\mathcal{T}$ be a terminology which contains $\text{elephant} \sqsubseteq \text{mammal}$.

Introduce a new primitive concept $\text{elephant}$.

Replace the specialization by the definition

$$\text{elephant} = \text{elephant} \sqcap \text{mammal}.$$ 

Do this for all specializations in $\mathcal{T}$ to obtain $\overline{\mathcal{T}}$.

$\overline{\mathcal{T}}$ is called normalization of $\mathcal{T}$.

Properties

- Every model of $\overline{\mathcal{T}}$ is a model for $\mathcal{T}$.
- For every model $I = (\mathcal{D}, \cdot^I)$ of $\mathcal{T}$ there is a model $\overline{I} = (\mathcal{D}, \cdot^{\overline{I}})$ of $\overline{\mathcal{T}}$ such that $I$ and $\overline{I}$ agree on the atomic concepts and rules occurring in $\mathcal{T}$. 
World Descriptions

- We want to describe a specific state of affairs given a terminology.
- Let $b, c, \ldots$ be constants.
- Assertions are:
  - Concept assertions $E(b)$.
  - Role assertions $R(b, c)$.
- A finite set of assertions is called world description or ABox.
An Example World Description

\[
\begin{align*}
\text{woman} &= \text{person} \sqcap \text{female} \\
\text{man} &= \text{person} \sqcap \neg \text{woman} \\
\text{mother} &= \text{woman} \sqcap \exists \text{hasChild}. \text{person} \\
\text{father} &= \text{man} \sqcap \exists \text{hasChild}. \text{person} \\
\text{parent} &= \text{father} \sqcup \text{mother} \\
\text{gndmother} &= \text{mother} \sqcap \exists \text{hasChild}. \text{parent} \\
\text{motherWoD} &= \text{mother} \sqcap \forall \text{hasChild}. \neg \text{woman}
\end{align*}
\]

\[
\begin{align*}
\text{motherWoD}(\text{mary}) & \quad \text{father}(\text{peter}) \\
\text{hasChild}(\text{mary}, \text{peter}) & \quad \text{hasChild}(\text{peter}, \text{harry}) \\
\text{hasChild}(\text{mary}, \text{paul}) &
\end{align*}
\]
Semantics – Assertions

Let $I = (\mathcal{D}, \cdot^I)$ be an interpretation.

- $[b]^I \in \mathcal{D}$ for each constant $b$.
- If $b$ and $c$ are different constants, then $[b]^I \neq [c]^I$.
- $[E(b)]^I = \top$ iff $[b]^I \in [E]^I$.
- $[R(b, c)]^I = \top$ iff $(a]^I, [b]^I) \in [R]^I$.

Let $\mathcal{T}$ be a TBox and $\mathcal{A}$ an ABox.

- $I \models \mathcal{T} \cup \mathcal{A}$ iff $I$ maps each element of $\mathcal{T} \cup \mathcal{A}$ to $\top$. 
Reasoning Tasks for Concepts

Let $\mathcal{T}$ be a Tbox and $E, F$ concepts.

**Satisfiability**
$E$ is satisfiable wrt $\mathcal{T}$ if there exists a model $I$ of $\mathcal{T}$ such that $[E]^I \neq \emptyset$.

**Subsumption**
$E$ is subsumed by $F$ wrt $\mathcal{T}$ if $[E]^I \subseteq [F]^I$ for every model $I$ of $\mathcal{T}$.
In this case we write $E \sqsubseteq_{\mathcal{T}} F$ or $\mathcal{T} \models E \sqsubseteq F$.

**Equivalence**
$E$ and $F$ are equivalent wrt $\mathcal{T}$ if $[E]^I = [F]^I$ for every model $I$ of $\mathcal{T}$.
In this case we write $E \equiv_{\mathcal{T}} F$ or $\mathcal{T} \models E \equiv F$.

**Disjointness**
$E$ and $F$ are disjoint wrt $\mathcal{T}$ if $[E]^I \cap [F]^I = \emptyset$ for every model $I$ of $\mathcal{T}$.

If $\mathcal{T} = \emptyset$ then we omit the qualification “wrt $\mathcal{T}$” and “$\mathcal{T}$”.
Examples – Reasoning Tasks for Concepts

Let $\mathcal{T}$ consist of the axioms:

- $\text{woman} = \text{person} \sqcap \text{female}$
- $\text{man} = \text{person} \sqcap \neg \text{woman}$
- $\text{mother} = \text{woman} \sqcap \exists \text{hasChild.person}$
- $\text{father} = \text{man} \sqcap \exists \text{hasChild.person}$
- $\text{parent} = \text{father} \sqcup \text{mother}$
- $\text{gndmother} = \text{mother} \sqcap \exists \text{hasChild.parent}$
- $\text{motherWoD} = \text{mother} \sqcap \forall \text{hasChild.} \neg \text{woman}$

Then:

- $\mathcal{T} \models \text{woman} \sqsubseteq \text{person}$.
- $\mathcal{T} \models \text{mother} \sqsubseteq \text{person}$.
- $\mathcal{T} \models \text{gndmother} \sqsubseteq \text{mother}$.
- $\text{woman}$ and $\text{man}$ are disjoint wrt $\mathcal{T}$.
- $\text{father}$ and $\text{mother}$ are disjoint wrt $\mathcal{T}$.
Reduction to Subsumption and to Unsatisfiability

Consider $\mathcal{ALC}$.

**Reduction to Subsumption** For concepts $E$ and $F$ we have:

- $E$ is unsatisfiable iff $E$ is subsumed by $\text{bot}$.
- $E$ and $F$ are equivalent iff $E$ is subsumed by $F$ and $F$ is subsumed by $E$.
- $E$ and $F$ are disjoint iff $E \sqcap F$ is subsumed by $\text{bot}$.

**Reduction to Unsatisfiability** For concepts $E$ and $F$ we have:

- $E$ is subsumed by $F$ iff $E \sqcap \neg F$ is unsatisfiable.
- $E$ and $F$ are equivalent iff both $E \sqcap \neg F$ and $F \sqcap \neg E$ are unsatisfiable.
- $E$ and $F$ are disjoint iff $E \sqcap F$ is unsatisfiable.

Hence, it suffices to develop algorithms that decide satisfiability.
Elimination of the TBox

Let $\mathcal{T}$ be an acyclic TBox and $\mathcal{T}'$ its expansion, and let $E$ be a concept.

The expansion of $E$ wrt $\mathcal{T}$ is the concept $E'$ obtained from $E$ by replacing each occurrence of a defined concept $A$ in $E$ by $F$, where $A = F \in \mathcal{T}'$.

Example The expansion of $\text{woman} \sqcap \text{man}$ wrt to our example TBox is

$$\text{person} \sqcap \text{female} \sqcap \text{person} \sqcap \neg (\text{person} \sqcap \text{female}).$$

Some facts:

- $\mathcal{T} \models E \equiv E'$.
- $E'$ is satisfiable wrt $\mathcal{T}$ iff $E'$ is satisfiable.
- $\mathcal{T} \models E \sqsubseteq F$ iff $\mathcal{T} \models E' \sqsubseteq F'$.
- $\mathcal{T} \models E \equiv F$ iff $\mathcal{T} \models E' \equiv F'$.
- $E$ and $F$ are disjoint wrt $\mathcal{T}$ iff $E'$ and $F'$ are disjoint.
Reasoning Tasks for ABoxes

- An ABox $\mathcal{A}$ is **satisfiable** wrt to a TBox $\mathcal{T}$, if there is a model for $\mathcal{T} \cup \mathcal{A}$.

- An ABox $\mathcal{A}$ is **satisfiable** if it is satisfiable wrt to the empty TBox.

- **Example**
  - Is $\{\text{father}(\text{mary}), \text{mother}(\text{mary})\}$ satisfiable?
  - Is $\{\text{father}(\text{mary}), \text{mother}(\text{mary})\}$ satisfiable wrt our example TBox?

- An assertion $\alpha$ is **entailed** by $\mathcal{A} \cup \mathcal{T}$, in symbols $\mathcal{A} \cup \mathcal{T} \models \alpha$, if every model for $\mathcal{A} \cup \mathcal{T}$ is also a model for $\alpha$.

- **Some facts**
  - $\mathcal{A} \cup \mathcal{T} \models \alpha$ iff $\mathcal{A} \cup \mathcal{T} \cup \{\neg \alpha\}$ is unsatisfiable.
  - $E$ is satisfiable iff $E(b)$ is satisfiable, where $b$ is an arbitrarily chosen constant.
Elimination of the TBox

► Let $\mathcal{T}$ be an acyclic TBox and $\mathcal{T}'$ its expansion.

► The expansion of an ABox $\mathcal{A}$ wrt $\mathcal{T}$ is the ABox $\mathcal{A}'$ obtained from $\mathcal{A}$ by replacing each occurrence of a defined concept $A$ in $\mathcal{A}$ by $F$, where $A = F \in \mathcal{T}'$.

► Let $\alpha'$ be the expansion of the assertion $\alpha$ wrt $\mathcal{T}$.

► Some facts

  ▶ $\mathcal{A}$ is satisfiable wrt $\mathcal{T}$ iff $\mathcal{A}'$ is satisfiable.
  ▶ $\mathcal{A} \cup \mathcal{T} \models \alpha$ iff $\mathcal{A}' \cup \{\neg \alpha'\}$ is unsatisfiable.
More Reasoning Tasks for ABoxes

- Let $\mathcal{T}$ be a TBox and $\mathcal{A}$ and ABox.

- **Retrieval problem**
  Given a concept $E$, find all constants $b$ such that $\mathcal{A} \cup \mathcal{T} \models E(b)$.

- **Realization problem**
  Given a constant $b$, find the most specific concepts $E$ such that $\mathcal{A} \cup \mathcal{T} \models E(b)$.
The Oedipus Example

- Some Greek mythology in a nutshell: Oedipus killed his father, married his mother Jocasta, and had children with her, among them Polyneikes. Polyneikes had also children, among them Thersandros. It is known that Thersandros did not kill his father.

- Let $A_{oe}$ consists of the following facts:

\[
\begin{align*}
\text{hasChild} & (jocasta, oedipus) \\
\text{hasChild} & (oedipus, polyneikes) \\
\text{hasChild} & (jocasta, polyneikes) \\
\text{hasChild} & (polyneikes, thersandros) \\
\text{patricide} & (oedipus) \\
\neg \text{patricide} & (thersandros)
\end{align*}
\]

- Does the following hold?

\[
A_{oe} \models (\exists \text{hasChild}. (\text{patricide} \land \exists \text{hasChild}. \neg \text{patricide})) (jocasta)
\]
A Tableau Based Satisfiability Algorithm for $\mathcal{ALC}$

- Let $\alpha$ be an assertion in negation normal form.
- Let $S_0 = \{\{\alpha\}\}$ ($S_0 = [\langle \alpha_0 \rangle]$).
- While one of the following rules is applicable to an element $A$ of $S_i$, apply it to obtain $S_{i+1}$.
  - If $A$ contains $(E_1 \cap E_2)(X)$, but does not contain both $E_1(X)$ and $E_2(X)$, then add $E_1(X)$ and $E_2(X)$ to $A$.
  - If $A$ contains $(E_1 \cup E_2)(X)$, but neither $E_1(X)$ nor $E_2(X)$, then replace $A$ by $A \cup \{E_1(X)\}$, $A \cup \{E_2(X)\}$.
  - If $A$ contains $(\exists R. E)(X)$, but there is no $Z$ with $\{R(X, Z), E(Z)\} \subseteq A$, then add $R(X, Y)$ and $E(Y)$ to $A$, where $Y$ is a constant not occurring in $A$.
  - If $A$ contains $(\forall R. E)(X)$ and $R(X, Y)$, but does not contain $E(Y)$, then add $E(Y)$ to $A$. 
Results

- Let $S = \{A_1, \ldots, A_k\}$ ($S = [A_1, \ldots, A_k]$).
- $S$ is satisfiable iff there is some $i$, $1 \leq i \leq k$, such that $A_k$ is satisfiable.
- Proposition Let $S'$ be obtained from $S$ by applying one of the transformation rules. $S$ is satisfiable iff $S'$ is satisfiable.
- Proposition There cannot be an infinite sequence $S_0, S_1, \ldots$
- An ABox $A$ contains a clash if one of the following occurs:
  - $bot(X) \in A$.
  - $\{A(X), \neg A(X)\} \subseteq A$ for some constant $X$ and some atomic concept $A$.
- Proposition $S$ is unsatisfiable iff all elements of $S$ contain a clash.
- Theorem It is decidable whether an $\mathcal{ALC}$-concept is satisfiable.
Example

Let $\mathcal{T}$ be our example TBox.

Does $\mathcal{T} \models \text{mother} \sqsubseteq \text{person}$ hold?

Recall $\text{mother} = ((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild.person}) \in \mathcal{T}'$.

Eliminating the TBox:
Does $\models ((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild.person}) \sqsubseteq \text{person}$ hold?

Reduction to concept unsatisfiability:
Is $((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild.person}) \sqcap \neg \text{person}$ unsatisfiable?

Reduction to ABox unsatisfiability:
Is $(((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild.person}) \sqcap \neg \text{person})(b)$ unsatisfiable?
Example – Continued

Applying the tableau based algorithm we obtain:

\[
S_0 = \{ \{((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild} \cdot \text{person}) \sqcap \neg \text{person})(b)\} \}
\]

\[
S_1 = \{ \{((\text{person} \sqcap \text{female}) \sqcap \exists \text{hasChild} \cdot \text{person})(b), \neg \text{person}(b)\} \}
\]

\[
S_2 = \{ \{\text{person}(b), \exists \text{hasChild} \cdot \text{person}(b), \neg \text{person}(b)\} \}
\]

\[
S_3 = \{ \{\text{person}(b), \text{hasChild}(b, c), \text{person}(c), \neg \text{person}(b)\} \}
\]

\[
S_4 = \{ \{\text{person}(b), \text{female}(b), \text{hasChild}(b, c), \text{person}(c), \neg \text{person}(b)\} \}
\]

- \(S_4\) contains the clash \{\text{person}(b), \neg \text{person}(b)\}.
- Hence, \(S_4\) is unsatisfiable.
- Hence, \(S_0\) is unsatisfiable.
- Hence, \(\mathcal{T} \models \text{mother} \sqsubseteq \text{person}\) does hold.
Fuzzy Description Logics

- There is a long debate concerning crisp vs. fuzzy relations.
- Fuzzy concepts can be found in [Tresp, Molitor: 1998] or $\mathcal{ALC}_F$ [Straccia: 2001].
The Fuzzy Description Logic $\mathcal{ALC}_F$

- **Concepts**
  
  $E, F \relax\rightarrow top$ \[ top^I(d) = 1 \text{ for all } d \in D \]
  
  $bot$ \[ bot^I(d) = 0 \text{ for all } d \in D \]
  
  $A$ \[ A^I : D \rightarrow [0, 1] \]
  
  $R$ \[ R^I : (D \times D) \rightarrow [0, 1] \]
  
  $E \sqcap F$ \[ (E \sqcap F)^I(d) = \min\{E^I(d), F^I(d)\} \]
  
  $E \sqcup F$ \[ (E \sqcup F)^I(d) = \max\{E^I(d), F^I(d)\} \]
  
  $\neg E$ \[ (\neg E)^I(d) = 1 - E^I(d) \]
  
  $\forall R.E$ \[ (\forall R.E)^I(d) = \inf_{d' \in D}\{\max\{1 - R^I(d, d'), E^I(d')\}\} \]
  
  $\exists R.C$ \[ (\exists R.C)^I(d) = \sup_{d' \in D}\{\min\{R^I(d, d'), E^I(d')\}\} \]

- **Terminological Axioms**
  
  $A \sqsubseteq E$ \[ A^I(d) \leq E^I(d) \text{ for all } d \in D \]
  
  $A = E$ \[ A^I(d) = E^I(d) \text{ for all } d \in D \]

- **Assertions**
  
  $\langle a : E \circ n \rangle$ \[ E^I([a]^I) \circ n \]
  
  $\langle (a, b) : R \circ n \rangle$ \[ R^I([a]^I, [b]^I) \circ n \]

$\langle a : E \circ n \rangle$ $(\circ \in \{<, \leq, \geq, >\})$

$\langle (a, b) : R \circ n \rangle$ $(n \in [0.1])$
Example

- Let $\mathcal{A}$ be the following Abox:
  - “If a customer is interested in an attribute, he wants a product $p$ that has that attribute” holds to a degree more or equal than 0.9:
    \[
    \langle p : \neg(\exists \text{attr}, \text{int}) \sqcup \text{wants} \geq 0.9 \rangle
    \]
  - “The customer is interested in technics” to $\geq 0.8$:
    \[
    \langle \text{tech} : \text{int} \geq 0.8 \rangle
    \]
  - “The product $p$ has attribute technics” to $\geq 0.7$:
    \[
    \langle (p, \text{tech}) : \text{attr} \geq 0.7 \rangle
    \]
- Does $\mathcal{A}$ entail that the customer wants product $p$ to a degree of at least 0.9?
  - $\mathcal{A} \models \langle p : \text{wants} \geq 0.9 \rangle$ iff $\mathcal{A} \cup \{\langle p : \text{wants} < 0.9 \rangle\}$ is unsatisfiable.
- The decision algorithm generates two completions, both of which contain a clash.
Example (Continued)

- Let \( A \) be the following Abox:
  - “If a customer is very interested in an attribute, he wants a product \( p \) that has that attribute” holds to a degree more or equal than 0.9:
    \[
    \langle p : \neg(\exists \text{attr}, \text{very int}) \sqcup \text{wants} \geq 0.9 \rangle
    \]
  - “The customer is interested in technics” to \( \geq 0.8 \):
    \[
    \langle \text{tech} : \text{int} \geq 0.8 \rangle
    \]
  - “The product \( p \) has attribute technics” to \( \geq 0.7 \):
    \[
    \langle (p, \text{tech}) : \text{attr} \geq 0.7 \rangle
    \]

- Does \( A \) entail that the customer wants product \( p \) to a degree of at least 0.9?
  \[
  A \models \langle p : \text{wants} \geq 0.9 \rangle \iff A \cup \{\langle p : \text{wants} < 0.9 \rangle\} \text{ is unsatisfiable.}
  \]
The Fuzzy Description Logic with Linear Hedges $\mathcal{ALC}_{FLH}$

► **Concepts**

**$E, F \rightarrow$**

- **$top$**: $[\text{top}]^I(d) = 1$ for all $d \in \mathcal{D}$
- **$bot$**: $[\text{bot}]^I(d) = 0$ for all $d \in \mathcal{D}$
- **$A$**: $[A]^I : \mathcal{D} \rightarrow [0, 1]$
- **$ME$**: $[ME]^I(d) = \eta_M([E]^I(d))$
- **$E \cap F$**: $[E \cap F]^I(d) = \min\{E^I(d), F^I(d)\}$
- **$E \cup F$**: $[E \cup F]^I(d) = \max\{E^I(d), F^I(d)\}$
- **$\neg E$**: $[\neg E]^I(d) = 1 - [E]^I(d)$
- **$\forall R. E$**: $[\forall R. E]^I(d) = \inf_{d'\in\mathcal{D}}\{\max\{1 - [R]^I(d, d'), [E]^I(d')\}\}$
- **$\exists R. C$**: $[\exists R. E]^I(d) = \sup_{d'\in\mathcal{D}}\{\min\{[R]^I(d, d'), [E]^I(d')\}\}$

where $\eta_M$ is a membership modifier or a linguistic hedge.

► **Roles**

**$R \rightarrow Q$**

- **$MR$**: $[MR]^I : (\mathcal{D} \times \mathcal{D}) \rightarrow [0, 1]$

$[MR]^I(d, d') = \eta_M([R]^I(d, d'))$
Hedges

\[ \Box \mathcal{ALC}_{FH} \]

\[ \beta = exponent(M), \text{ where } M \text{ is a modifier.} \]

\[ \Box \text{Exponential hedges} \]

\[ \eta_M(x) = x^\beta, \text{ where } x \in [0, 1]. \]

\[ \Box \mathcal{ALC}_{FLH} \]

\[ \beta = exponent(M), \text{ where } M \text{ is a modifier.} \]

\[ \Box \text{Linear hedges} \]

\[ \eta_M(x) = \eta_\beta(x) = \begin{cases} \frac{1}{\beta} x & \text{if } x \leq \frac{\beta}{\beta+1}, \\ 1 + \beta(x - 1) & \text{otherwise.} \end{cases} \]

\[ \Box \text{Example} \]

\[ \text{porsche-cayenne-turbo} = \text{suv} \sqcap \text{more race-car} \sqcap \text{less all-terrain-vehicle} \]

\[ \text{race-car} = \text{car} \sqcap \exists speed. \text{very very high} \]
Discussion

- $\mathcal{ALC}$, $\mathcal{ALC}_F$, $\mathcal{ALC}_{FH}$, $\mathcal{ALC}_{FLH}$.
- We would like to build in trace rules.
- We are investing hedge algebras.
- We are looking for real world examples.
- We are aiming at a prototypical implementation.
Literature

