The Calculus of Structures

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The Calculus of Structures

Outline of the Course

1. What is the calculus of structures?
2. Classical Logic
   - atomicity, locality
3. Linear Logic
   - modularity
4. System SBV
   - coalition (and process algebras)
What is the calculus of structures?

It's a step back from the sequent calculus.

Diagram:

- Computation
  - Proof-search
  - Proof-normalization
  - Better?
- Sequent calculus
- Calculus of structures
- New properties:
  - Atomicity
  - Locality
  - Modularity
- Distributed computation

Do we get a better proof theory?
1. What is the calculus of structures? 2/4

The sequent calculus is very committed to trees 1/2

Example 1. "Additive" conjunction

The formula tree shapes the proof tree

Example 2. "Multiplicative" conjunction

The formula tree induces an unwanted tree (in proof-search)
Example 3  Cut elimination

\[
\begin{align*}
\text{cut} & \quad \Pi_1, \Gamma, \Delta \\
\text{cut} & \quad \Pi_2, \Gamma, \Delta \\
\text{cut} & \quad \Pi_3, \Gamma, \Delta \\
\end{align*}
\]

The formula tree decides the order of reductions
1. **What is the calculus of structures?**

Trees are unfriendly to distributed computation.

- **Example** Suppose that:
  - atoms are processors: a, b, c, d
  - communication flows through the tree structure

```
  1
 / \  
2   3
/ \ / \ 
a b c d
```

The communication workload of 1 is four times that of 2 and 3.

- Main connectives create an asymmetry

- Step back: in the calculus of structures, there are no main connectives
What is the calculus of structures?

There are no main connectives

Example 1 "Additive" conjunction

\[ \vdash (A \land C) \land (A \lor C) \]
\[ \vdash (A \land B) \lor C \]

Example 2 "Multiplicative" conjunction

\[ \vdash (A \times C) \otimes B \]
\[ \vdash (A \otimes B) \otimes C \]

Inference rules can be applied deep inside formulæ.

There is a new top-down symmetry.

What happens to the subformule property?
**Example 1**  "Additive" conjunction

Rule

\[
p \frac{\Sigma \{ (AVC) \land (BVC) \} S}{\Sigma \{ (A \land B) \lor C \} S}
\]


\[
p \frac{( (AVC) \land (BVC) \land D ) \lor E}{((A \land B) \lor C) \land D \lor E}
\]


**Example 2**  "Multiplicative" conjunction

Rule

\[
p \frac{\Sigma \{ (AVC) \otimes B \} S}{\Sigma \{ (A \otimes B) \lor C \} S}
\]

\[
p \frac{( (AVC) \otimes (B \otimes D) )}{((A \otimes B) \otimes C) \otimes D}
\]
Inference rules can be applied deep inside formulae.

- Inference rule p:

- The hole in $S \otimes 3$ does not appear inside a negation.

- Rule $p$ corresponds to $T \rightarrow R$
1. What is the calculus of structures?

Structures

- Atoms are positive or negative: \(e, b, c, \ldots, \overline{e}, \overline{b}, \overline{c}, \ldots\)

- Structures \(P, Q, R, S, T, U, \ldots\) are:

\[
S ::= \text{atoms} \quad \text{or} \quad a
\]
\[
\text{disjunctions} \quad \left[ S, \ldots, S \right]
\]
\[
\text{conjunctions} \quad \left(S, \ldots, S\right)
\]
\[
\text{other relations} \quad \langle S, \ldots, S \rangle \quad | \quad \ldots
\]
\[
\text{units} \quad \text{or} \quad t, f, 1, \overline{1}, \ldots
\]
\[
\text{modelled structures} \quad ?S \quad !S \quad \ldots
\]
\[
\text{quantified structures} \quad \exists x.S \quad \forall x. S \quad \ldots
\]
\[
\text{negated structures} \quad \overline{S}
\]
Equations are imposed over structures:

- Commutativity (not always): \([R,T] = [T,R]\)
- Associativity (always): \(\langle R; \langle T; u \rangle \rangle = \langle R; T; u \rangle\)
- De Morgan (always!): \(\overline{[R,T]} = (\overline{R}, \overline{T})\)
- Content level closure: \(R = T \Rightarrow S\{R\} = S\{T\}\)

Notation: Braces are dropped when unnecessary. Example:

- \(S[R,T]\) instead of \(S\{[R,T]\}\)
What is the calculus of structures?

There is a new top-down symmetry

If

\[ \text{SRT3} \]

is a rule, corresponding to

\[ T \rightarrow R \]

then

\[ \text{SRT3} \]

is also a rule, corresponding to

\[ R \rightarrow T \]
Example

In linear logic:

\[
\begin{array}{c}
\text{Proof}\downarrow \\
S \Downarrow !([R,T]) \\
\hline \\
S ![R,T]
\end{array}
\]

Corresponds to

\(!([R,T]) \rightarrow ![R,T]\)

corresponds to

\[
\begin{array}{c}
\text{Proof}\uparrow \\
S \Uparrow ?([R,T]) \\
\hline \\
S ![R,T]
\end{array}
\]

\(!([R,T]) \rightarrow ![R,T]\)
There is a new top-down symmetry

- Derivations $(\Delta)$ are chains of instances of inference rules

\[ \vdash U \]
\[ \vdash T \]
\[ \vdash R \]

There is a top-down symmetry. Example

\[ \vdash R \]
\[ \vdash T \]
\[ \vdash U \]

is a valid derivation
What is the calculus of structures?

What happens to the subformula property?

- Morally, it still holds if we design rules carefully. Examples in

\[
\frac{S([R, U], T)}{S([R, T], U)}
\]

premise and conclusion are made of the same pieces

- Rules can still be finitary, either upwards, or downwards, or both

- Being finitary does not depend on having main connectives
Do we get a better proof theory?

- We have some chances because:
  - we established the main connective idea
  - we are free to apply rules deeply
  - then we have more freedom
  - we also have a new symmetry!
  - we should see proofs in more detail

- But:
  - we have to be careful in designing systems!
    (we shouldn’t abuse freedom)
  - it’s still not clear whether we can do some good distributed computation
Recipe for a good system

1. Choose disjunction and conjunction and make identity and cut.

Example: linear logic

- \([R, T]\) stands for \(R \otimes T\)
- \((R, T)\) stands for \(R \otimes T\)

2. Establish

\[
\frac{S[R, R]}{S[R, R]}
\]

interaction down or identity

\[
\frac{S(R, R)}{S[R, 3]}
\]

interaction up or cut

3. This is your interaction fragment
Recipe for a good system

- Take each couple of dual logical relations, for example:
  - \( \{R,T\} \) stands for \( R \circ T \)
  - \( (R,T) \) stands for \( R \wedge T \)
- and create the rules

\[
\begin{align*}
\frac{S([R,U],[T,V])}{S([R,T],[U,V])} & \quad \frac{S([R,T],[U,V])}{S([R,U],[T,V])} \\
\frac{S([\forall n. R],[T])}{S([\forall n. R],[\exists n. T])} & \quad \frac{S([\exists n. R],[\forall n. T])}{S([\exists n. R],[\forall n. T])}
\end{align*}
\]
- or, for example

\[
\begin{align*}
\frac{S([\forall n. R],[\exists n. T])}{S([\forall n. R],[\exists n. T])} & \quad \frac{S([\exists n. R],[\forall n. T])}{S([\exists n. R],[\forall n. T])}
\end{align*}
\]
- This is your core structure fragment
- Add the non-core structure fragment
A one-sided system into the calculus of structures

One-sided (Gentzen-Schütte)

A system for classical logic in the calculus of structures (the "neit" system)

\[
\begin{align*}
\text{id} & \frac{}{A, \vec{A}} \\
\text{VL} & \frac{\Gamma, A}{\Gamma, A, B} \\
\land & \frac{\Gamma, A, B}{\Gamma, A \land B} \\
\omega t & \frac{\Gamma, A}{\Gamma, A, A} \\
\text{vt} & \frac{\Gamma, A}{\Gamma, A, \vec{A}} \\
\text{id} & \frac{S}{(S, [A, \vec{A}])} \\
\text{VR} & \frac{\Gamma, B}{\Gamma, A, B} \\
\text{V} & \frac{(S, \Gamma)}{(S, [\Gamma, A])} \\
\land & \frac{(S, [\Gamma, A], [\Gamma, B])}{(S, [\Gamma, (A, B)])} \\
\land & \frac{(S, [\Gamma, A, A])}{(S, [\Gamma, A])} \\
\land & \frac{(S, [\Gamma, (A, B)])}{(S, [\Gamma, A])} \\
\end{align*}
\]
**Equations**

\[
\begin{align*}
[r] &= (r) = r \\
[r, \overline{r}] &= [\overline{r}, r] \\
(r, \overline{r}) &= (\overline{r}, r) \\
\overline{r} &= r \\
[r, [\overline{r}, \overline{v}], \overline{v}] &= [r, \overline{r}, [\overline{v}, v]] \\
(r', [\overline{r}, \overline{v}], [\overline{r}, v]) &= (r', [\overline{r}, v], [\overline{r}, \overline{v}])
\end{align*}
\]

If \( r = t \) then \( 5[r3] = 5[\overline{r}] \)

\[
[r, t] = r - (r, t)
\]

\[
\bar{t} = f \\
\bar{f} = t
\]

**Example** Prove \( (A \lor B) \Rightarrow A \Rightarrow A \Rightarrow ((A \lor v)^{\lor} \lor A) \)

\[
\begin{align*}
\begin{array}{c}
\vdash \overline{\overline{A}} \lor A \\
\vdash \overline{A} \lor B, A \\
\vdash (\overline{A} \lor B) \land \overline{A}, A \\
\vdash ((\overline{A} \lor B) \land \overline{A}) \lor A, A \\
\vdash ((\overline{A} \lor B) \land \overline{A}) \lor A, \overline{A} \\
\vdash ((\overline{A} \lor B) \land \overline{A}) \lor v, \overline{A}
\end{array}
\end{align*}
\]
The calculus of structures generalizes the one-sided sequent calculus.

- It is trivial and uninteresting to port a system in the one-sided sequent calculus to the calculus of structures.

- The translation works like this:

  \[
  \begin{align*}
  \pi' & \vdash \pi'' \\
  E, \ldots, E_h & \vdash E' \quad E' \\
  & \vdash E'' \quad E'' \\
  & \vdash F \\
  \end{align*}
  \]

  \[
  \begin{align*}
  \pi' + \pi'' & \vdash (E, \ldots, E_h, E', E'') \\
  \pi & \vdash (E, \ldots, E_h, E) \\
  \pi & \vdash F \\
  \end{align*}
  \]

- Symmetry is not exploited!

- Deepness is not exploited!

- Can we do better than the sequent calculus?
Classical logic

A deep, symmetric system

- Let's apply our recipe!
- We keep the equations we have already

Interaction

\[
\text{it } \frac{S[t3]}{S[R, \bar{R}]} \quad \text{it } \frac{S(R, \bar{R})}{S[t3]}
\]

Core structure

\[
\text{sl } \frac{S([R, U], [T, V])}{S[(R, T), U, V]} \quad \text{st } \frac{S([R, T], U, V)}{S[(R, U), (T, V)]}
\]

Non-core structure (here we have to be creative)

\[
\text{wl } \frac{S[t3]}{S[R]} \quad \text{wl } \frac{S[R]_3}{S[t3]}
\]

\[
\text{ct } \frac{S[R, R]}{S[R]} \quad \text{ct } \frac{S[R]_3}{S(R, R)}
\]
A deep, symmetric system

- Definition A system \( \mathcal{S} \) is a set of inference rules.

- Definition A rule \( \alpha \) is strongly admissible for a system \( \mathcal{S} \) if \( \alpha \notin \mathcal{S} \) and for every instance \( \frac{T}{R} \) there is a derivation \( \frac{\mathcal{S}}{R} \).

- Definition This rule is called switch: \[
\frac{S([R,U],T)}{S([R,T],U)}
\]

- Proposition \( S! \) and \( S^! \) are strongly admissible for \( S \)

  Proof

  \[
  \begin{align*}
  & S([R,U],[T,V]) \\
  & S([R,U],T,V) \\
  & S([R,T],U,V) \\
  & S([R,T],U,V) \\
  & S([R,U],T,V) \\
  & S([R,T],U,V)
  \end{align*}
  \]

- Remark Switch is self-dual

- Remark \( S \) is a special case both of \( S! \) and \( S^! \)
We have a system, let's call it CLC.

Is this classical logic? Yes: let's see.

Remark: $\exists, \forall, \otimes, s3$ (and $\exists_0, s_3$) is multiplicative linear logic.
Theorem

Every derivation in \( \text{GSL}_p \) can be transformed into a derivation in \( \text{CLC} \), and if it is cut-free, it remains cut-free.

Proof

\( \text{CLC} \) is more general than the usual system we saw already. (just pay attention to contraction in the rule \( \text{A} \) and notice that

\[
\begin{align*}
\frac{(S, [\Gamma, A], [\Delta, A])}{(S, [\Gamma, [\Delta, A]], [\Delta, A])}, \\
\frac{(S, [\Delta, [\Gamma, A]], [\Delta, A])}{(S, [\Gamma, [\Delta, A]], [\Delta, A])}, \\
\frac{(S, [\Gamma, \Delta, (A, A)])}{(S, [\Gamma, \Delta])}
\end{align*}
\]

Then, \( \text{CLC} \) is classical logic, because every rule is sound.

Is there any use for \( \text{W} \) and \( \text{CS} \)?
2. Classical logic

A deep, symmetric system

- What about cut elimination?

- Idea: let's exploit the sequent calculus

- Theorem: Every derivation in CLC can be transformed into a derivation in GSMp

**Proof**

\[
\frac{\Delta \vdash \Sigma T3}{\text{cut}} \quad \frac{\Sigma T3}{\text{cut}} \quad \frac{\Sigma T3, \Sigma R3}{\Sigma R3}
\]

This is easily done for each \( p \).
A deep, symmetric system

- Let's break the symmetry!

- **Definition** A proof is a derivation whose topmost structure is (equivalent to) $\vdash$

- **Definition** An inference rule $\gamma$ is (weakly) admissible for a system $\mathcal{S}$ if $\gamma \& \mathcal{S}$ and for every proof $\Gamma \vdash R \mathcal{S}$ there exists a proof $\Gamma \vdash R$

- **Theorem** It is admissible for $\{ \text{ib, s, wb, cb}\}$ (and there is an algorithmic transformation for it)

**Proof**

\[
\begin{array}{ccc}
\text{calculus of structures} & \text{sequent calculus} \\
\Delta & \rightarrow & \varepsilon(\Delta) \text{ (lots of cuts)} \\
\downarrow & \text{cut elimination} & \Downarrow \\
(\text{no cuts}) & \varepsilon'(\Delta') & \Delta'
\end{array}
\]
2. Classical logic

A deep, symmetric system

- Do we have a better system than classical logic in the sequent calculus?
  Perhaps, but still ...

- Do we have a better, or interesting, cut elimination procedure?
  Well ...

- Symmetry still is not fully exploited!

- Deeper still is not fully exploited!
Atomicity

Consider:

\[
\frac{S[t, b]}
{S[(R, T), R, T]}
\]

The it's became "smaller", so they eventually can be replaced by

\[
\frac{S[t, b]}
{S[e, e]}
\]

This rule is called atomic interaction.

Theorem: It is strongly admissible for \( \exists a : b, c \)

Nothing unexpected!
Atomicity

- Consider

\[\begin{align*}
\frac{S([T], R, \bar{R}, \bar{T})}{S([T], R, \bar{R}, \bar{T})} \\
\frac{S([T], R, \bar{R}, \bar{T})}{S([T], R, \bar{R}, \bar{T})} \\
\frac{S([T], R, \bar{R}, \bar{T})}{S([T], R, \bar{R}, \bar{T})}
\end{align*}\]

The \( \{R, T\} \)'s, too, become "smaller"; we can replace them by

\[\frac{S([T], R, \bar{R}, \bar{T})}{S([T], R, \bar{R}, \bar{T})} \]

This rule is called atomic co-interaction

- Theorem if \( \{R, T\} \) is strongly admissible for \( \{a \top, s\} \)

- This property, due to symmetry, we can exploit!
Atomicity of counteraction (cut)

- Consequences:
  - a simpler cut elimination proof
  - decomposition theorems

- Curiosities:
  - a different relation between cut, subformula property, and finitarguency
  - a simple consistency proof
Finitaryness

- In the sequent calculus finitaryness (going up) corresponds to the subformula property.

Example

\[
\begin{align*}
\to & \vdash A, B \\
\to & \vdash A, A, B
\end{align*}
\]

- Finitary
- A and B are subformulas of A, B

\[
\begin{align*}
\not\vdash & \vdash A, A, \neg A, \neg A \\
\vdash & \vdash A
\end{align*}
\]

- Non-finitary
- A is not necessarily a subformula of the conclusion

- In the calculus of structures there is no subformula property, but still all inference rules for classical logic are finitary (going up), except for

\[
\begin{align*}
\text{wt} & \frac{S \{ e, e \} }{S \{ e \}} \\
\text{int} & \frac{S \{ R, \bar{R} \} }{S \{ R \}}
\end{align*}
\]

(or \( \text{dit } \frac{S \{ e, e \} }{S \{ e \}} \))
Finiteryness

• Rules in the core are always finitary!
  (They just "reshuffle" logical relations)

• Conules in the non-core up fragment are always strongly admissible for their duals, plus switch and intersections:

  \[
  \begin{align*}
  \text{it} & \quad \frac{S(T)}{S(T, [R, \overline{R}])} \\
  \text{gt} & \quad \frac{S(T, [R, \overline{R}])}{S(T, [R, \overline{T}])} \\
  \text{s} & \quad \frac{S(T, [R, \overline{T}])}{S(T, [R, (T, T)])} \\
  \text{it} & \quad \frac{S(T, [R, (T, T)])}{S[R, T, 3]} \\
  \end{align*}
  \]

• Then the only infinitary rule we are left with is

  \[
  \text{at } \frac{S(2, 3)}{S[4, 3]}
  \]
2. Classical logic

Finitaryness

- Consider the finitary atomic coinduction rule:

\[
\text{Fait} \xrightarrow{S(a, \bar{a})} \text{S} \quad \text{where } a \text{ or } \bar{a} \text{ appears in } S \text{ and } 3
\]

- It is easy to eliminate all sit instances that are not fait instances, in proofs:

\[
\begin{array}{c}
\text{bottommost sit} \\
\text{which is not an sit} \\
\text{fait}
\end{array} \quad \xrightarrow{S(a, \bar{a})} \text{S} \\
\text{fait} \quad \text{R}
\]

replace here all \( \bar{a} \)'s with \( t \) and all \( a \)'s with \( f \): the proof remains valid!

proceed inductively upwards in the proof.

- Theorem Replacing sit by fait does not affect provability

- Finitaryness does not morally depend on full-blown cut elimination!
A simple consistency proof

Theorem: Propositional classical logic is consistent
Proof: We cannot get $t \parallel f$ when using falsity.

Theorem: If $R$ is provable then $\overline{R}$ is not provable.
Proof: Suppose we have

\[
\begin{align*}
\pi_1 \parallel R \\
R \\
\text{and} \\n\pi_2 \parallel \overline{R}
\end{align*}
\]

Then we make $\pi_1, \pi_2 \parallel (R, \overline{R})$ and then we flip it:

\[
\begin{align*}
\parallel \overline{R} \\
\parallel f
\end{align*}
\]

Then we can make:

\[
\begin{align*}
\parallel \overline{R} \\
\parallel f
\end{align*}
\]

absurd.
2 Classical logic

Exploiting deepness

- The following rule is called medial:

\[\frac{S[(R,0),(T,V)]}{S[R,T],[U,V]}\]

- Medial is self-dual

- Look at

\[\frac{S[p,p,a,a]}{S[p,p]} \quad \frac{S[(p,a),(p,a)]}{S[p,p],[a,a]}\]

\[\frac{S[p,p]}{S[p,a]} \quad \frac{S[p,p],a}{S[p,a]}\]

By medial, contractions get "smaller"

- The following rules are called atomic contraction and atomic cocontraction:

\[\frac{S[e,e]}{S[e,e]} \quad \frac{S[e,e]}{S[e,e]}\]

- Theorem cb is strongly admissible for cb,m3, and dually
Deepness is essential for getting atomic contraction

In the sequent calculus, it is impossible to get atomic contraction

By the way, weakening is easily reduced to atomic form:

\[
\frac{S(f, f)}{S(f, f)} \quad \frac{S(f, q)}{S(f, q)}
\]

\[
\frac{S[f, f]}{S[f, f]} \quad \frac{S[f, q]}{S[f, q]}
\]

and obviously for conweakening
This is classical logic.
Locality

- Let's call locality the property of a rule requiring bounded effort to be applied.

Example: switch

![Diagram: Two structures with labels R, T, U, and [R,T),U] on the left and [R,U),T] on the right.]

- Locality depends on the representation

- Atomicity can be a special form of locality

- There still is much to do for distributed computation (but look at relational fields)

- Applications in complexity?
Why cut elimination is different than in the sequent calculus?

Because in the sequent calculus the main connective "drives" the reduction:

\[
\frac{\frac{}{\Gamma, A} \quad \frac{}{\Gamma, B}}{\frac{}{\Gamma, \Lambda B} \quad \frac{}{\Gamma, \Lambda B}} \quad \frac{\frac{}{\Gamma, \Lambda B} \quad \frac{}{\Gamma, \Lambda B}}{\frac{}{\Gamma, \Lambda B}}
\]

\[
\frac{\frac{}{\Gamma, A} \quad \frac{}{\Gamma, A}}{\frac{}{\Gamma, A} \quad \frac{}{\Gamma, A}}
\]
Cut elimination

- In the calculus of structures:

\[
\begin{align*}
\frac{}{
\frac{(e, [d, (b, c, [\bar{e}, \bar{b}, \bar{c}])])}{[d, (e, b, c, [\bar{e}, \bar{b}, \bar{c}])]} = \\
\text{if } \frac{S(R, T, [\bar{R}, \bar{T}])}{S\{f\}}}
\end{align*}
\]

\[R = (e, b), \quad T = c, \quad S = [d, 3, 3]\]

What are we supposed to do??

- Freedom has a price

- Atomicity helps a lot!
Theorem \( \text{cut} \uparrow \) is admissible.

Proof

1. Transform cuts into shallow cuts:

\[
\text{cut} \uparrow \frac{[S, (e, \bar{e})]}{S}
\]

2. Permute up super cuts:

\[
\text{cut} \uparrow \frac{(s_i, s_2)}{[(s'_1, u.e), (s'_2, u.e)]}
\]

where \( u.e = (e, \ldots, e) \) \( n \) times.

and \( S' \) is obtained from \( S \) by replacing some \( e \)'s by \( f \);

and \( S'_2 \) is obtained from \( S_2 \) by replacing some \( \bar{e} \)'s by \( f \).
Decompositions

- Theorems

- For every $T \vdash_{skx} \text{there is a}$
  $R$

- For every $T \vdash_{skx} \text{there is a}$
  $R$

- One cannot do these things in the sequent calculus

- We start seeing some modularity
Is there any use for weakening and contraction?

Yes:

- We saw already for getting and (but that use was trivial)

- In interpolation theorems:
  
  It is always possible to generate derivations such that, if \( T \rightarrow R \), then

\[
\begin{array}{c}
T \\
\parallel \downarrow \text{growing} \\
U \leftarrow \text{interpolant} \quad T \rightarrow U \rightarrow R \\
\parallel \downarrow \text{growing} \\
R
\end{array}
\]
Linear logic

Multiplicative exponential linear logic

System SELS

down up

\[ \frac{S(c, e)}{S(c, c)} \quad \frac{S(c, e)}{S(c, c)} \]

Intersection structure

\[ S([c(c,0,1), p] ) \]

\[ S([c(c,1,0), n] ) \]

... deciding equations, especially

\[ ?[c] = [c] \]

\[ ![c] = ![c] \]
Linear logic

Multiplicative exponential linear logic

- Interactions are atomic
- Promotion is local!
- Absorption (i.e., contraction) is not atomic
- Modularity starts to manifest itself: each of \( a \cdot \top, \quad p \cdot \top, \quad u \cdot \top \) and \( b_1 \) is admissible for the down fragment and can be shown admissible independently (to a certain extent)
- So, there are \( 2^4 = 16 \) equivalent systems whose properties are known
Theorem  For every $T \vdash R$

There is a sequence of $\{ \lambda \}$

Proof Difficult!
Full linear logic

- We apply all our techniques and get:

- A system, called SCLS, with 34 rules, 16 of which in the up-on-down fragment (all admissible, of course): so we have $2^{16} = 65,536$ equivalent systems

- All rules are local (or atomic), including contradictions

- All rules follow our recipe + medial + contraction + weakening, so the system is big but very uniform
### Full Linear Logic

<table>
<thead>
<tr>
<th>Structure</th>
<th>Interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S([R, T], U)$</td>
<td>$S([R, T], U)$</td>
</tr>
<tr>
<td>$S([R, U], [T, V])$</td>
<td>$S([R, T], [U, V])$</td>
</tr>
<tr>
<td>$S([R, T])$</td>
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<tr>
<td>$S([0])$</td>
<td>$S([a, a])$</td>
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### System SLLS
It always holds. How do we prove it?

MLL: splitting

NELL: decomposition + splitting

NALL: splitting

LL: by translation to the sequent calculus
Idea

- CCS is a language for distributed computation where

\[ a \cdot b \mid \bar{a} \cdot \bar{b} \rightarrow 0 \]

- Can we make a logic out of this?

- If so, we want \( \bar{a} \cdot b = \bar{b} \cdot a \)

- Then \( \cdot \) is a non-commutative self-dual logical relation

- Problem: getting this in the sequent calculus is very difficult (let's say impossible, see later)
Recipe!

- Ingredients:
  2 commutative dual logical relations
  1 non-commutative self-dual logical relation
  1 self-dual unit common to all relations

- Recipe:
  Just create an intersection and a core structure fragment (everything is multiplicative, for now)

- We get a very simple system whose proof theory is extremely intricate

- The system is atomic and local
The system

- Rules:

- Equations:

  Commutativity:
  \[
  [\bar{a}, \bar{a}] = [\bar{a}, \bar{a}]
  \]
  \[
  (\bar{a}, \bar{a}) = (\bar{a}, \bar{a})
  \]

  Associativity:
  \[
  [\bar{a}, [\bar{a}, \bar{a}]] = [\bar{a}, [\bar{a}, \bar{a}]]
  \]
  \[
  ([\bar{a}, [\bar{a}, \bar{a}]] = [\bar{a}, [\bar{a}, \bar{a}]]
  \]
  \[
  (\bar{a}, (\bar{a}, \bar{a})) = (\bar{a}, (\bar{a}, \bar{a}))
  \]
  \[
  (\bar{a}, (\bar{a}, \bar{a})) = (\bar{a}, (\bar{a}, \bar{a}))
  \]
  \[
  (\bar{a}; (\bar{a}; \bar{a})) = (\bar{a}; (\bar{a}; \bar{a}))
  \]
  \[
  (\bar{a}; (\bar{a}; \bar{a})) = (\bar{a}; (\bar{a}; \bar{a}))
  \]

  Content clause:
  \[
  \text{if } R = T \text{ then } S[R] = S[T]
  \]

- Unit:

  \[
  R = [R, 0] = (R, 0) = (R; 0) = [0; R]
  \]

- Negation:

  \[
  \bar{a} = \bar{a}
  \]
  \[
  [\bar{a}, [\bar{a}, \bar{a}]] = [\bar{a}, [\bar{a}, \bar{a}]]
  \]
  \[
  ([\bar{a}, [\bar{a}, \bar{a}]] = [\bar{a}, [\bar{a}, \bar{a}]]
  \]
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  \[
  (\bar{a}; (\bar{a}; \bar{a})) = (\bar{a}; (\bar{a}; \bar{a}))
  \]

  Singletons:
  \[
  [R] = [R] = [R] = R
  \]
The idea comes from the sequent calculus.
• **Definition** \( BV = 2 d h, s, q b \)

• **Theorem (Splitting)**

  1. If \( \mathcal{F} \) then \([2, 3, <s_1, s_2>]\)

  \( s < R, T > \)

  \( s \in 3 \)

  \( [R, s, t] \)

  \( [T, s, t] \)

  2. If \( \mathcal{F} \) then \([2, 3, s_1, s_2]\)

  \( s (R, T) \)

  \( s \in 3 \)

  \( [R, s, t] \)

  \( [T, s, t] \)

**Proof** Complex, but uniform
System SBV

Cut elimination by splitting

1. Theorem $\alpha_I^P$ is admissible for $BV$
   
   Proof Splitting

2. Theorem $\gamma^B$ is admissible for $BV$
   
   Proof Splitting

3. SBV and $BV$ (and $BV \cup \{I, P\}$ and $BV \cup \{I, P, R\}$) are equivalent
Decomposition

Theorem

If \( T \) is SBV then \( R \)

Proof Permutations

\[
\begin{align*}
T & \leftarrow (a, b) \\
T' & \leftarrow \text{core of SBV} = \xi, \eta, \zeta \\
R' & \leftarrow (a, b) \\
R & \leftarrow R'
\end{align*}
\]
Intuitive representation of SBV structures

\(<a; [b, (c, \langle d; e \rangle)]>\)
SBV cannot be expressed in the sequent calculus.

System SBV
SBV cannot be expressed in the sequent calculus
SBV cannot be expressed in the sequent calculus

- Theorem $S_1, S_2, \ldots$ are all provable in SBV if and only if one starts reasoning from the looks

Proof Use relational fields semantics

- Theorem There is no system in the (normal) sequent calculus which is equivalent to SBV

Proof Given any sequent system, produce a structure $S_k$ whose look is deeper than the depth of the sequent system
The calculus of structures

Do we get a better proof theory?
Can we do better than the sequent calculus?

We observe:

- atomicity
- locality
- modularity:
  - in the rules
  - in decompositions
  - in cut elimination arguments
- we easily define logics that 'challenge' the sequent calculus