Satisfiability Checking using Semantic Trees

- Semantic Trees
- Relating Interpretations and Semantic Trees
- Satisfiability Checking
Key Characteristics of Semantic Trees

- Optimization of truth table method.
- Stepwise partitioning of interpretations (through branching).
- Usually conceived for formulas in clausal form.
Semantic Trees

- A semantic tree for a finite set $S$ of clauses is a binary tree satisfying the following conditions:
  - Edges are labeled by literals.
  - If $A$ or $\neg A$ are labels of edges, then $A$ occurs in $S$.
  - For each pair of edges with the same parent node $N$, one edge is labeled by $A$ while the other one is labeled by $\neg A$.
  - There is no branch in $S$, where both, $A$ and $\neg A$, occur as labels for edges.
  - In each branch $A$ or $\neg A$ occur at most once as label.
  - Each leaf node $N$ may be labeled by a clause $C \in S$ provided that the complements of all literals in $C$ occur among the labels of the branch from $N$ to the root.

- Note that semantic trees are finite because there occur only finitely many propositional variables $A$ in a finite set $S$ of clauses and each branch contains at most one occurrence of $A$ or $\neg A$. 
Closed Semantic Trees

- Given a finite set $S$ of clauses.
  - A branch of a semantic tree for $S$ is said to be closed iff its leaf node is labeled by a clause from $S$.
  - Otherwise a branch is said to be open.
  - A semantic tree for $S$ is is said to be closed iff each of its branches is closed.
Semantic Tree Generation

► Given a finite set $S$ of clauses.

► Generate a root node.

► Apply the following rules to some open leaf node $N$:
  
  ▶ **Reduction** If the branch from $N$ to the root of the tree can be closed, then close it by labeling $N$ with the corresponding clause from $S$.

  ▶ **Extension** If there occurs a variable $A$ in $S$ such that neither $A$ nor $\neg A$ is used as a label on the branch from $N$ to the root then add two child nodes to $N$ and label the new edges with $A$ and $\neg A$. 
Example

Consider the following set of clauses:

\[ S = \{C_1 : q \lor \neg r, C_2 : q \lor r, C_3 : \neg p \lor \neg q, C_4 : p \lor \neg r, C_5 : p \lor \neg q \lor r\} \]

This semantic tree is closed.

\( S \) is unsatisfiable.
Saturated Semantic Trees

- A branch in a semantic tree is said to be of maximal length iff for each propositional variable $A$ occurring in $S$ the branch contains
  - either an edge labeled by $A$
  - or an edge labeled by $\neg A$.

- A semantic tree for a finite set $S$ of clauses is said to be saturated iff each of its branches is
  - either closed
  - or is open, cannot be closed and has maximal length.

- In a saturated semantic tree neither reduction nor extension steps are possible.

- Each closed semantic tree is saturated.
Another Example

Consider the following set of clauses:

\[ S' = \{ C_2 : q \lor r, C_3 : \neg p \lor \neg q, C_4 : p \lor \neg r, C_5 : p \lor \neg q \lor r \} \]

This semantic tree is saturated and contains a branch which cannot be closed.

\( S' \) is satisfiable.
Relating Branches to Interpretations

- Let $S$ be a finite set of clauses.
- Let $N$ be a leaf node in a semantic tree for $S$.
- Let $M$ be the set of labels on the branch from $N$ to the root.
- $M$ represents a partial interpretation $I_M$ as follows:

\[
A^{I_M} = \begin{cases} 
\top & \text{if } A \in M, \\
\bot & \text{if } \neg A \in M, \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]
Relating Models to Semantic Trees

Lemma 1 Let $S$ be a finite set of clauses and $T$ a semantic tree for $S$. Then, for each model $I$ of $S$ exists exactly one branch $B$ of $T$ such that

- for all labels $L$ on $B$ we find $L^I = \top$ and
- $B$ cannot be closed by a clause from $S$.

Proof

- If $[] \in S$, then $S$ has no models and the claim holds.
- Assume that either $S = \emptyset$ or each clause in $S$ contains at least one literal.
- Case: $T$ consists just of the root node:
  - Consequently, the root node is the only branch $B$ of $T$.
  - $B$ has the empty set of labels.
    - Hence, $L^I = \top$ holds for all labels $L$ on $B$ and each model $I$ of $S$.
  - $B$ is open, because there is no clause in $S$ which might close it.
Proof of Lemma 1 (Continued)

- Otherwise: $T$ has more nodes than just the root node.
  - For a given model $I$ of $S$ we can select a branch $B$ of $T$ by stepping down the tree while always following the edge whose label $L$ is true under $I$. These selections are deterministic and therefore the branch $B$ is uniquely determined.
  - Let us now assume that $B$ is closed.
    - Then exists a clause $C \in S$ with $\overline{L}$ among the labels of $B$ for all literals $L \in C$.
    - Because the labels of $B$ are all true under $I$, $[\overline{L}]^I = \top$ holds.
    - We conclude that $L^I = \bot$ for all literals $L$ in $C$.
    - Consequently, $I$ is not a model of $S$. Contradiction.
Relating Semantic Trees to Models

Lemma 2  Let $S$ be a finite set of clauses and $T$ a semantic tree for $S$. Then, if $B$ is a branch of $T$

- which is of maximal length and
- which cannot be closed by a clause from $S$

then exists a model $I$ of $S$ with $L^I = \top$ for all $L \in M$ where $M$ is the (maximal) set of labels on the branch $B$.

Proof

- Construct $I_M$.
- We find: $L^{IM} = \top$ for all $L \in M$.
- Because $B$ cannot be closed, this implies that for all $C \in S$ exists a literal $L_C \in C$ such that $\overline{L_C} \notin M$.
- Because $B$ is of maximal length, this implies $L_C \in M$ for all $C \in S$, and consequently $L^{IM}_C = \top$.
- This implies $C^{IM} = \top$ for all $C \in S$, i.e., $I_M$ is a model of $S$.  ■
Semantic Trees and Satisfiability Testing

- Given a (finite) set $S$ of clauses.
  - (Immediately from Lemma 1)
    - If $S$ is satisfiable, then each semantic tree for $S$ has a branch which cannot be closed by a clause from $S$.
    - If $S$ is satisfiable, then every saturated semantic tree for $S$ is open.
  - (Immediately from Lemma 2)
    - If there exists a semantic tree for $S$ with a branch of maximal length which cannot be closed by a clause from $S$, then $S$ is satisfiable.
    - If there exists an open saturated semantic tree for $S$, then $S$ is satisfiable.
Correctness of Unsatisfiability Checking

▸ **Theorem 3** If there exists a closed semantic tree $T$ for a finite set $S$ of clauses, then $S$ is unsatisfiable.

▸ **Proof**

▸ Let us assume that $S$ is satisfiable.
▸ Then exists a model $I$ of $S$.
▸ From Lemma 1 follows that there exists an open branch in $T$.
▸ Contradiction to $T$ being closed.
Completeness of Unsatisfiability Checking

**Theorem 4** If a finite set $S$ of clauses is unsatisfiable, then exists a closed semantic tree for $S$.

**Proof**

- Let us assume that there exists no closed semantic tree for $S$.
- Then each saturated semantic tree for $S$ has an open branch $B$ of maximal length.
- According to Lemma 2 exists a model $I$ of $S$.
- Contradiction to unsatisfiability.
Relationship to the Truth Table Method

- Branches correspond to rows in the truth table.

- Advantage over the truth table method:
  Closed branches don’t need to have maximal length.

- Results similar to Lemma 1 and Lemma 2 hold for the truth table method and allow to prove its correctness and completeness.