

# The Fuzzy Description Logic $ALC_{FH}$ with Hedge Algebras as Concept Modifiers

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## Abstract

In this paper we present the fuzzy description logic  $ALC_{FH}$  introduced, where primitive concepts are modified by means of hedges taken from hedge algebras.  $ALC_{FH}$  is strictly more expressive than Fuzzy- $ALC$  defined in [11]. We show that given a linearly ordered set of hedges primitive concepts can be modified to any desired degree by prefixing them with appropriate chains of hedges. Furthermore, we define a decision procedure for the unsatisfiability problem in  $ALC_{FH}$ , and discuss knowledge base expansion when using terminologies, truth bounds, expressivity as well as complexity issues. We extend [8] by allowing modifiers on non-primitive concepts and extending the satisfiability procedure to handle concept definitions.

**Key words:** fuzzy logic, hedge algebras, description logic

## 1 Introduction

In many areas of computer science and artificial intelligence logic has been accepted as the mathematical foundation for knowledge representation and reasoning. Beyond that it has been recognized that the logic itself can be used as computation unifying in an almost ideal way declarative and operational semantics. Among the many applications description logics have been particularly successful (see e.g. [2, 4]).

The main task of a description logic is to fix a terminology by describing concepts and their relations, and to provide (decidable) key services for reasoning wrt this terminology like, for example, subsumption and unsatisfiability testing.

In all applications of description logics that we are aware of, except [11] and [12], concepts are crisp unary relations, i.e., an object may or may not be an element of a particular concept. On the other hand, in many real-world applications like, for example, intelligent e-commerce information is often vague and imprecise. We may observe that a customer is interested in technical aspects, whereas he or she is not interested in design issues. Fuzzy set theory introduced by Lotfi A. Zadeh (see e.g. [13]) provides an ability to denote non-crisp concepts, i.e., an object may belong to a certain degree (typically a real number from the interval  $[0, 1]$ ) to a particular relation. For instance, the semantic content of customer Robert being interested in technical aspects may be described by means of a statement like “Robert is interested in technical aspects” and establishing that this sentence has a degree, or truth-value, of 0.8.

Humans typically use linguistic adverbs like “very”, “more or less” etc. to distinguish, for example, between a customer who is interested in technical details and one who is very interested in these details. In [1] Zadeh introduces so-called linguistic hedges modifying the shape of a fuzzy set by transforming it into another. For instance, he introduced the operator  $CON$  (for “contraction”) that maps the membership function  $\mu_A$  of a fuzzy set  $A$  to a membership function  $\mu_{very A}$  that has high degrees of membership only for those elements of the do-

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main, that belong “very much” to  $A$ . In other words, the degree of membership for elements that already had a low degree of membership in  $A$  is decreased, whereas the degree of membership for elements with a high degree of membership is left high. Technically, Zadeh achieves this by simply raising the degree of membership to the  $\beta$ -th power, where  $\beta > 1$  is a constant which can be fixed by the application. For example,  $\mu_{\text{very } A}(u) = \mu_A(u)^2$ . This technical solution calls for the introduction of an opposite operator  $DIL$  (for dilation) that maps the degree of membership to the  $\beta'$ -th power, where  $0 < \beta' < 1$ , thereby strengthening the degree of membership of elements with small degree. One should note that these hedges can be concatenated. For example, if we want to consider elements that belong to  $A$  “very very much”, then  $\mu_A$  can be transformed into its  $\beta^2$ -th power by applying the operator  $CON$  twice:  $CON CON A$ .

Because the gap between  $A$  and  $CON A$  in this setting is quite large, it seems a good idea to refine this setting. In many human languages there is almost a continuum of phrases like “more or less”, “much less”, “possibly rather” and so forth expressing different levels of emphasis. The so called *hedge algebras* introduced in [7, 6] give an algebraic structure that formally defines such hedges and structures their relationships. They have been applied to fuzzy logic in various ways (see e.g. [5]), but to our knowledge, hedge algebras have not been applied as concept modifiers in a description logic.

In this paper we apply linear ordered hedge algebras as concept modifiers in the framework of a fuzzy description logic. The paper is organized as follows: In the following Section 2 we recall basic notions from fuzzy description logics and the structure of hedges. In particular, we define an extended fuzzy description logic with hedges  $ALC_{FH}$ . Concept modifiers are formally introduced in Section 3, and we describe an algorithm that maps each concept modifier to an operator on membership functions. A decision procedure for the unsatisfiability problem of  $ALC_{FH}$  is presented in Section 4. Expressivity and complexity issues are discussed in Sections 5 and 6. The decision procedure is extended to knowledge bases with terminologies in Section 7. Section 8 disputes the problem of determining subsumptions. Finally, we discuss our findings in Section 9.

## 2 Logical Basics

### 2.1 Fuzzy Description Logic

Concepts are expressions that collect properties of a set of individuals, described—among others—by means of roles. From a first order logical viewpoint, concepts can be seen as unary predicates, whereas roles are interpreted as binary predicates. A concept can be build out of primitive concepts, roles, modifiers and combinations of other concepts (see Table 1). For the fuzzy extension, a concept  $C$ , rather than being interpreted as a classical set, will be represented as a fuzzy set and, thus, concepts become imprecise. Consequently, a statement like “ $a$  is  $C$ ”, where  $a$  is an individual, will have a truth-value in  $[0, 1]$  denoting the degree of the membership of  $a$  in the fuzzy set  $C$ .

$C, D \rightarrow$	$A$	(primitive concept)
	$\top$	(top concept)
	$\perp$	(bottom concept)
	$\neg C$	(negation)
	$M(C)$	(concept modification)
	$C \sqcap D$	(concept conjunction)
	$C \sqcup D$	(concept conjunction)
	$\forall R.C$	(universal quantification)
	$\exists R.C$	(existential quantification)

Table 1: The language  $ALC_{FH}$  of the description logic  $ALC$  extended by fuzzy hedges. As a notational convention,  $C$  and  $D$  denote concepts,  $A$  primitive concepts,  $R$  roles and  $M$  concept modifiers, all of them possibly with subscripts. As a syntactical convenience, the parentheses around  $C$  may be omitted in  $M(C)$  iff  $C$  is a primitive concept.

The semantics is based on the notion of an interpretation. An interpretation is a pair  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  consisting of a non-empty set  $\Delta^{\mathcal{I}}$  (called the domain) and an interpretation function  $\cdot^{\mathcal{I}}$  mapping individuals to elements of  $\Delta^{\mathcal{I}}$ , concepts  $C$  to a membership function  $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$ , and roles  $R$  to a membership function  $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$ . Therefore, if  $d \in \Delta^{\mathcal{I}}$  is an object of the domain  $\Delta^{\mathcal{I}}$ , then  $C^{\mathcal{I}}(d)$  is the degree of  $d$  being an element of the fuzzy concept  $C$  under  $\mathcal{I}$ . Roles are interpreted in the same way. The interpretation for complex concepts is defined in Table 2.<sup>1</sup>

Two concepts  $C$  and  $D$  are said to be *equivalent* (denoted by  $C \cong D$ ) when  $C^{\mathcal{I}} = D^{\mathcal{I}}$  for all interpretations  $\mathcal{I}$ . For instance,  $\top \cong \neg \perp$ ,

<sup>1</sup>The similarity of the definitions of many of the operators defined in Table 2 with fuzzy set operations [9] is intended.

$$\begin{aligned}
A^{\mathcal{I}} &: \Delta^{\mathcal{I}} \rightarrow [0, 1] \\
R^{\mathcal{I}} &: \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1] \\
\top^{\mathcal{I}}(d) &= 1 \text{ for all } d \in \Delta^{\mathcal{I}} \\
\perp^{\mathcal{I}}(d) &= 0 \text{ for all } d \in \Delta^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}}(d) &= \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) &= \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) &= 1 - C^{\mathcal{I}}(d) \\
(M(C))^{\mathcal{I}}(d) &= \eta_M(C^{\mathcal{I}}(d)) \\
(\forall R.C)^{\mathcal{I}}(d) &= \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) &= \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}
\end{aligned}$$

Table 2: Interpretations of  $ALC_{FH}$ .  $\eta_M$  is a membership modifier discussed later.

$$\begin{aligned}
(C \sqcap D) &\cong \neg(\neg C \sqcup \neg D) \\
\text{and } (\forall R.C) &\cong \neg(\exists R.\neg C).
\end{aligned}$$

A *fuzzy assertion* (denoted by  $\psi$ ) is one of  $\langle \alpha > n \rangle$ ,  $\langle \alpha \geq n \rangle$ ,  $\langle \alpha \leq n \rangle$ ,  $\langle \alpha < n \rangle$  and  $\langle \alpha = n \rangle$  where  $\alpha$  is an expression of type  $a : C$  (“ $a$  is in  $C$ ”),  $(a, b) : R$  (“ $(a, b)$  is in  $R$ ”) and  $n \in [0, 1]$ . From a semantical viewpoint, a fuzzy assertion  $\langle \alpha \geq n \rangle$  constrains the fuzzy truth-value<sup>2</sup> of  $\alpha$  to be greater or equal to  $n$ . Formally, an interpretation  $\mathcal{I}$  satisfies a fuzzy assertion  $\langle a : C \geq n \rangle$  (resp.  $\langle (a, b) : R \geq n \rangle$ ) iff  $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$  (resp.  $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$ ). The operators  $>$ ,  $=$ ,  $<$  and  $\leq$  are interpreted in the same way. Two fuzzy assertions  $\psi_1$  and  $\psi_2$  are said to be *equivalent* (denoted by  $\psi_1 \cong \psi_2$ ) iff they are satisfied by the same set of interpretations.

A *fuzzy terminological axiom* is either a *fuzzy concept specialization* of the form  $A \leq C$  or a *fuzzy concept definition* of the form  $A := C$ . A specialization denotes the “more specific than”-relation between concepts, and a definition denotes the equivalence of concepts. A fuzzy interpretation  $\mathcal{I}$  satisfies a fuzzy concept specialization  $A \leq C$  iff for all  $d \in \Delta^{\mathcal{I}}$  we find  $A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d)$ , whereas  $\mathcal{I}$  satisfies a fuzzy concept definition  $A := C$  iff for all  $d \in \Delta^{\mathcal{I}}$  we find  $A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d)$ .

A *fuzzy knowledge base*  $\Sigma$  is the union of a finite set  $\Sigma_A$  of fuzzy assertions and a finite set  $\Sigma_T$  of fuzzy terminological axioms. An interpretation  $\mathcal{I}$  *satisfies* (or is a *model* of) a fuzzy knowledge base  $\Sigma$  iff  $\mathcal{I}$  satisfies each element of  $\Sigma$ . A fuzzy know-

<sup>2</sup>This fuzzy truth value corresponds to the membership value of  $a$  in  $C$  when  $\alpha = a : C$ , or of  $(a, b)$  in  $R$  when  $\alpha = (a, b) : R$ .

ledge base *entails* a fuzzy assertion  $\psi$  (denoted by  $\Sigma \models \psi$ ) iff every model of  $\Sigma$  also satisfies  $\psi$ . Furthermore, let  $C$  and  $D$  be two concepts,  $D$  *fuzzy subsumes*  $C$  with respect to  $\Sigma_T$  (denoted by  $C \preceq_{\Sigma_T} D$ ) iff for every model  $\mathcal{I}$  of  $\Sigma$  and for all  $d \in \Delta^{\mathcal{I}}$  we find  $C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$ .

The problem of determining whether  $\Sigma \models \psi$  is called *entailment problem*, whether  $C \preceq_{\Sigma_T} D$  is called *fuzzy subsumption problem*, and whether  $\Sigma$  is satisfiable is called *satisfiability problem*.

## 2.2 The Structure of Hedges

In fuzzy set theory, hedges were defined as unary operators which act on fuzzy sets, and form new fuzzy sets [1], for instance,  $\mu_{\text{very}A}(u) = \mu_A(u)^2$ , where  $u$  an element of the universe  $U$  and  $\mu_A$  is the membership function of the fuzzy set  $A$ . Starting from a set of hedges  $H$ , for example  $\{\text{very}, \text{mol}, \text{rather}, \text{possibly}\}$ ,<sup>3</sup> a hedge algebra defines an ordering relation  $\leq$  on linguistic terms. In the work of Nguyen Cat Ho *et al.* [7, 5] linguistic terms are defined as chains of hedges applied to so called generators, whose truth value is modified by the hedges. In the context of description logics the generators are the primitive concepts. In this paper, we assume that  $H$  is linearly ordered such that any two hedges are comparable. In the literature hedge algebras with partially ordered sets of hedges are also studied by extending refined hedge algebras with distributive properties. In principle, the method for mapping linguistic terms to fuzzy sets seems to be applicable in case of partially ordered sets of hedges as well if a linearization of the order is considered.

In contrast to the hedges introduced by Zadeh, a chain of hedges has a meaning only taken as a whole: each new hedge put in front *refines* the meaning of the others. Consider, for instance, the hedge *very*. Intuitively, a chain containing several *very* strengthens the meaning of a concept:  $\varepsilon \leq \text{very} \leq \text{veryvery} \leq \dots$ , where  $\varepsilon$  is the empty sequence of hedges. On the other hand, *rather* should weaken<sup>4</sup> a concept:  $\text{rather} \leq \varepsilon$ . Intuitively, however,  $(\text{very})^n \text{rather} A$  for any  $n > 0$  is still a weaker concept than  $A$  itself, i.e.

$$\begin{aligned}
\text{rather} &\leq \text{very rather} \leq \text{veryvery rather} \\
&\leq \dots \leq \varepsilon \leq \text{very} \ .
\end{aligned}$$

However, while  $DILA$  is a weaker concept than  $A$ ,  $CONCONDILA$  is a stronger concept than  $A$ .

<sup>3</sup>*mol* is an abbreviation for *more or less*.

<sup>4</sup>“Weaken” in the sense of subsumption: a concept  $D$  is weaker than a concept  $C$  iff it subsumes  $C$ .

This causes a discrepancy between the utilization of linguistic terms made in natural language and the mathematical representation obtained using *CON* and *DIL*.

Furthermore, a hedge can be both positive and negative wrt different others, i.e., it increase or decrease their meaning, respectively. "Very", for example, is positive to "very" when it strengthens the effect:  $x \leq \text{very } x \leq \text{very very } x$ . On the other hand, it can be at the same time negative to other hedges like "possibly", when it reduces its effect:  $\text{possibly } x \leq \text{very possibly } x \leq x$ , and, thus,  $\text{very possibly } x$  is less different from the original meaning  $x$  than  $\text{possibly } x$ .

An extensive discussion of hedge algebras, as given in [7, 5], is out of the scope of this paper. Thus, we will only introduce those notions and properties that will be used in the remainder of the paper. Two linguistic terms  $u$  and  $v$  are *independent* if neither of them can be formed by putting some more hedges in front of the other. To capture this property formally, we introduce  $H(x)$  as the set of linguistic terms that can be formed by prefixing  $x$  with chains of hedges:  $H(x) = \{\delta x \mid \delta \text{ is a chain of hedges}\}$ . Then,  $u$  and  $v$  are independent if  $u \notin H(v)$  and  $v \notin H(u)$ . Consequently we have that  $x \notin H(v)$  for all  $x \in H(u)$ . For two different hedges  $h$  and  $k$ , prefixing other hedges  $h'$  and  $k'$  to  $h$  and  $k$  resp. can not change the semantical ordering relationship between  $hx$  and  $kx$ , i.e.,  $h'hx < k'kx$  iff  $hx < kx$ .

In the next section we will define concept manipulators using hedge algebras.

### 3 Concept Modifiers

Let us consider the set  $H = \{h_1, h_2, \dots, h_p\}$  of hedges, among which each element of  $H$  is either positive or negative wrt all primitive concepts and all other hedges including itself. Let  $\Omega$  be the set of all primitive concepts. We define  $sign : H \times (H \cup \Omega) \rightarrow \{-1, 1\}$  with

$$sign(h_i, h^*) = \begin{cases} -1 & \text{if } h_i \text{ is negative wrt } h^*, \\ 1 & \text{if } h_i \text{ is positive wrt } h^*. \end{cases}$$

where  $h_i \in H$ ,  $h^* \in H \cup \Omega$ . The mapping  $sign$  tells us whether  $h_i$  increases or decreases the meaning of  $h^*$ , if  $h^* \in \Omega$ , or  $h^* \delta A$ , where  $h^*$  is a hedge,  $A$  is a primitive concept and  $\delta$  is any chain of hedges.

In  $ALC_{FH}$  we introduce concept modifiers  $M$  in form of a chain of hedges  $M = k_q k_{q-1} \dots k_1$

with  $k_i \in H$ , for all  $i = \overline{1, q}$ . In this section we will define a mapping  $\eta_M : [0, 1] \rightarrow [0, 1]$  that gives each hedge chain a meaning within the interpretation defined in Table 2. We will follow the idea of Zadeh to use power functions for this purpose.

**Definition 1.** A membership modifier is an exponential function  $\eta : [0, 1] \rightarrow [0, 1]$  with  $\eta(x) = x^\beta$ , where  $\beta > 0$ .

So we have to calculate an exponent  $\beta$  for each sequence of hedges. In the following, we will present an algorithm *exponent* which calculates  $\beta$  for each hedge chain from a giving hedge set  $H$  representing a membership modifier based on following ideas:

- We extend the idea behind Zadeh's operators *CON* and *DIL* discussed in the introduction in that we keep the idea of taking a power function with exponents  $\beta > 1$  for positive and  $0 < \beta < 1$  for negative hedges, but vary the exponents depending on the hedge. For instance when someone is 25 years, if the degree of being "young" is 0.8, then the degree of being "very young" might be  $0.8^2 = 0.64$ , whereas "more or less young" may have the degree of  $0.8^{1/2} = 0.894$ .
- Membership modifiers should preserve the independence property of hedges. I.e., for all  $h, h', k, k' \in H$  we should find that if  $h\delta < k\delta$  then  $h'h\delta < k'k\delta$ , where  $\delta$  is a chain of hedges. In other words,  $h'$  and  $k'$  can not change the semantic ordering between  $hx$  and  $kx$ . As we will show, this property carries over to the exponent, and thus, the modified concepts. Note that this is different from Zadeh's proposal of *CON* and *DIL*: they are commutative:  $CON DIL A = DIL CON A$ , whereas we have  $\text{very rather} < \text{rather very}$  if  $\text{rather} < \text{very}$ .
- The set  $H^* = \{\delta \mid \delta \text{ is a chain of hedges}\}$  of all membership modifiers should be mapped dense into the interval  $(0, \infty)$ . I.e. we should be able to approximate each value in  $(0, \infty)$  to any required precision by an  $exponent(\delta)$  such that  $\delta \in H^*$ . In case of  $\beta$  approaches 0, the correlative membership modifier may change membership degrees of all individuals to 1, and in inverse case, if  $\beta$  comes close to  $+\infty$ , then all membership degrees approximate 0 (see Figure 1).

Based on the above ideas, we favor in this paper an approach to construct membership modifiers in

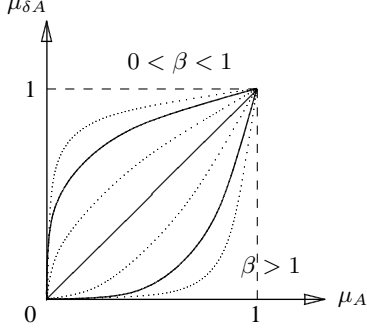


Figure 1: The membership modifier of  $\delta A$ .

which the greatest positive modifier (e.g. "very") behaves like the "traditional" operator  $CON$ . I.e. we choose a value  $upper > 1$  as exponent for  $CON$ , such that  $very A = CON A = A^{upper}$ ,  $very very A = CON CON A = A^{upper^2}$ , and so forth. Likewise, we choose a value  $0 < lower < 1$  as exponent for  $DIL$ , and treat the greatest negative modifier (e.g. "less") like  $DIL$ :  $less A = DIL A = A^{lower}$ ,  $very less A = DIL DIL A = A^{lower^2}$ . The values  $upper$  and  $lower$  can be chosen arbitrarily in the range  $upper > 1$  and  $0 < lower < 1$  as parameters for the algorithm below to build a family of concept modifiers.

To simplify the computing process we assume that  $H$  is linearly ordered, i.e., each two hedges are comparable, and restrict ourselves to hedge sets  $H$  of an even cardinality  $2p$ , and sets of positive hedges and negative hedges of the same size  $p$ , i.e., we can represent  $H$  as  $H = \{h_1^-, h_2^-, \dots, h_p^-, h_1^+, h_2^+, \dots, h_p^+\}$ , with positive hedges  $h_1^+ < h_2^+ < \dots < h_p^+$  and negative hedges  $h_1^- < h_2^- < \dots < h_p^-$ . To avoid intermingling between positive or negative hedges, we restrict the mapping  $sign$  such that for a given  $h^*$  either all positive hedges  $h_i^+$  increase the meaning of  $h^*$  and all negative hedges  $h_i^-$  decrease the meaning, i.e.,  $sign(h_i^+, h^*) = 1$  and  $sign(h_i^-, h^*) = -1$ , or the other way around, i.e., for  $i = \overline{1, p}$  we have that  $sign(h_i^+, h^*) = -1$  and  $sign(h_i^-, h^*) = 1$ . For the purpose of this paper we restrict  $sign$  such that  $sign(h_i^+, A) = 1$  and  $sign(h_i^-, A) = -1$  for all  $A \in \Omega$  and  $i = \overline{1, p}$ . Thus, the calculated exponents are independent of the concept they are calculated for.

The algorithm of calculating  $\beta$  for a chain  $M = k_q k_{q-1} \dots k_1$  of hedges is described in Figure 2. In the  $i$ -th step,  $sig$  indicates that  $k_i$  increases (by  $sig = 1$ ) or decreases (by  $sig = -1$ ) the value of  $\beta_{i-1}$ ;  $\beta_i$  will be the exponent for  $k_i k_{i-1} \dots k_1$ ; and  $(lo_i, up_i)$  defines the interval

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Algorithm *exponent*

**Constants:** the values  $upper > 1$ ,  $0 < lower < 1$ , the mapping  $sign : H \times (H \cup \Omega) \rightarrow \{-1, 1\}$

**Input:** Concept modifier  $M = k_q k_{q-1} \dots k_1$  with  $k_i \in H$  for all  $i = \overline{1, q}$

**Output:** The exponent  $\beta$

```

{
  up0 = upper ;   lo0 = lower ;   β0 = 1 ;
  mpos = 1 ;   mneg = 1 ;   sig = sign(k1, A) ;
  for i = 1 to q do {
    compute the index j such that
      ki = hj- or ki = hj+ ;
    if i > 1 then sig = sig × sign(ki, ki-1) ;
    if sig = 1 then {
      βi = βi-1 +  $\frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j - 1)$  ;
      upi = βi-1 +  $\frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j)$  ;
      loi = βi-1 +  $\frac{up_{i-1} - \beta_{i-1}}{2p} \times (2j - 2)$  ;
      mneg = 0 ;
      if mpos = 1 and j = p then {
        βi = upperi ; upi = upperi+1 ;
      }
      else mpos = 0 ;
    }
    else {
      mpos = 0 ;
      βi = βi-1 -  $\frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j - 1)$  ;
      upi = βi-1 -  $\frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j - 2)$  ;
      loi = βi-1 -  $\frac{\beta_{i-1} - lo_{i-1}}{2p} \times (2j)$  ;
      if mneg = 1 and j = p then {
        βi = loweri ; loi = loweri+1 ;
      }
      else mneg = 0 ;
    }
  }
}
return(βq) ;

```

Figure 2: The algorithm to calculate the exponent  $\beta$  of membership modifier  $M$ .

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from which the exponent of  $\delta k_i k_{i-1} \dots k_1$  can be obtained, where  $\delta$  is any chain of hedges. The calculation starts from  $(lo_0, up_0) = (lower, upper)$ , and this interval is successively shrunk depending on  $k_i$ . In case of most changing positive modifier (called the greatest positive modifier) like "very very ... very" we assign  $\beta_i = upper^i$  and  $up_i = upper^{i+1}$ , to extend the interval for  $\delta k_i k_{i-1} \dots k_1$  to  $(lo_i, +\infty)$ . In this case  $mpos$  remains at 1 until the first  $k_i$  with  $k_i \neq h_q^+$  is encountered. Then  $mpos$  is changed to 0, and the interval is not extended further, but subdivided as usual.  $mneg$  is dealt with in the inverse manner such that the exponent of  $\delta k_i k_{i-1} \dots k_1$  can receive values in  $(0, up_i)$ .

In the following we will prove some properties of the algorithm. In the proofs we will often need to discuss what happens in the main loop, whose behavior is different depending on the sequence  $k_i k_{i-1} \dots k_1$ . To save space, we will define four cases that will be referred to in the proofs by their number. Let us denote  $\Gamma_1 = \frac{up_{i-1} - \beta_{i-1}}{2p}$  and  $\Gamma_2 = \frac{\beta_{i-1} - lo_{i-1}}{2p}$ . By induction it is easy to see that  $\Gamma_1 > 0$  and  $\Gamma_2 > 0$  for all  $i$ , and we distinguish the cases:

(I)  $k_i$  is positive wrt  $k_{i-1} \dots k_1$  and  $k_i k_{i-1} \dots k_1$  is not the greatest positive modifier. Thus:

$$\begin{aligned}\beta_i &= \beta_{i-1} + \Gamma_1 \times (2j - 1); \\ up_i &= \beta_{i-1} + \Gamma_1 \times (2j); \\ lo_i &= \beta_{i-1} + \Gamma_1 \times (2j - 2).\end{aligned}$$

(II)  $k_i$  is positive wrt  $k_{i-1} \dots k_1$  and  $k_i k_{i-1} \dots k_1$  is the greatest positive modifier. Thus:

$$\begin{aligned}\beta_i &= upper^i; \\ up_i &= upper^{i+1}; \\ lo_i &= \beta_{i-1} + \Gamma_1 \times (2p - 2).\end{aligned}$$

(III)  $k_i$  is negative wrt  $k_{i-1} \dots k_1$  and  $k_i k_{i-1} \dots k_1$  is not the greatest negative modifier. Thus:

$$\begin{aligned}\beta_i &= \beta_{i-1} - \Gamma_2 \times (2j - 1); \\ up_i &= \beta_{i-1} - \Gamma_2 \times (2j - 2); \\ lo_i &= \beta_{i-1} - \Gamma_2 \times (2j).\end{aligned}$$

(IV)  $k_i$  is negative wrt  $k_{i-1} \dots k_1$  and  $k_i k_{i-1} \dots k_1$  is the greatest negative modifier. Thus:

$$\begin{aligned}\beta_i &= lower^i; \\ lo_i &= lower^{i+1}; \\ up_i &= \beta_{i-1} - \Gamma_2 \times (2p - 2).\end{aligned}$$

Before turning to the properties of this algorithm we would like to illustrate it by means of an example.

### Example 1.

Consider  $H = \{very, mol\}$  with  $p = 1$ ,  $upper = 2$ ,  $lower = 0.5$  and  $sign$  as follows:

	$A$	$very$	$mol$
$very$	1	1	-1
$mol$	-1	-1	1

Applying the algorithm we obtain:

	$sig$	$mpos$ $mneg$	$lo$	$\beta$	$up$
$very$	1	1 0	1	2	4
$mol$	-1	0 1	0.25	0.5	1
$very\ very$	1	1 0	2	4	8
$mol\ very$	-1	0 0	1	1.5	2
$very\ mol$	1	0 0	0.5	0.75	1
$mol\ mol$	-1	0 1	0.125	0.25	0.5

Let  $M = k_q k_{q-1} \dots k_1$  be a concept modifier, the corresponding values are denoted by  $lo_M$ ,  $\beta_M$  and  $up_M$ .  $\beta_M$  is  $\beta_q$ , the output of the Algorithm, and  $lo_M$ ,  $up_M$  are  $lo_q$ ,  $up_q$ , which are calculated in the  $q$ -th step, respectively. For examination of the independence property we need the following lemma.

**Lemma 1.** For any concept modifier  $M$  we find:  
 $lo_M < \beta_M < up_M$ .

*Proof.* It is easy to prove in the cases (I), (III), (II):  $\beta_M < up_M$  and (IV):  $lo_M < \beta_M$ . We have to consider the case (II), whether  $lo_M < \beta_M$  is satisfied. The case (IV) is similar.

Because  $2p - 2 < 2p$  we also obtain  $\frac{2p-2}{2p} < 1$ . Therefore  $lo_M = \beta_{q-1} + (up_{q-1} - \beta_{q-1}) \frac{2p-2}{2p} < \beta_{q-1} + (up_{q-1} - \beta_{q-1}) \times 1 = \beta_{q-1} + up_{q-1} - \beta_{q-1} = upper^q$  because  $k_{q-1} \dots k_1$  is also the greatest positive modifier of the length  $q - 1$ . So  $lo_M < \beta_M = upper^q$ .  $\square$

**Lemma 2.** For all  $k \in H$  we have:

$$lo_M \leq \beta_{kM} \leq up_M.$$

*Proof.* By case analysis it is easy to deduce the following constraints:

$lo_M < \beta_M \leq lo_{kM} < \beta_{kM} < up_{kM} \leq up_M$  in case (I),

$lo_M < \beta_M \leq lo_{kM} < \beta_{kM} = up_M < up_{kM}$  in case (II),

$lo_M \leq lo_{kM} < \beta_{kM} < up_{kM} \leq \beta_M < up_M$  in case (III),

$lo_{kM} < lo_M = \beta_{kM} < up_{kM} \leq \beta_M < up_M$  in case (IV).

Thus,  $lo_M \leq \beta_{kM} \leq up_M$  holds in all cases.  $\square$

Therefore we have the independence property of concept modifiers:

**Theorem 1.** Let  $M_1 = \delta_1 h k_q \dots k_1$  and  $M_2 = \delta_2 h' k_q \dots k_1$ , with  $h, h', k_i \in H$ ,  $i = \overline{1, q}$ , be two concept modifiers with the exponents  $\beta_1, \beta_2$ . Let  $\beta, \beta'$  be the exponents of  $h k_q \dots k_1$  and  $h' k_q \dots k_1$ . Then, we have that  $\beta_1 < \beta_2$  holds if  $\beta < \beta'$ .

*Proof.* Let  $M = k_q \dots k_1$ . Suppose  $\delta_1 = h_r \dots h_1$ . Since  $\delta_1 h M$  is not the greatest positive modifier (because  $\beta < \beta'$ ), we only need to pay attention to the cases (I), (III) and (IV). There we always find  $\beta_{hM} < up_{hM} \leq up_M$ . Thus,  $\beta_1 = \beta_{h_r \dots h_1 h M} < up_{h_r \dots h_1 h M} \leq up_{h_{r-1} \dots h_1 h M} \leq \dots \leq up_{h_1 h M} \leq up_{hM}$ . Similarly, we can obtain:  $lo_{h'M} < \beta_2$ .

Now we prove that  $up_{hM} \leq lo_{h'M}$ .

Assuming  $\beta < \beta'$ , we have to distinguish three cases:

1.  $h$  and  $h'$  are positive wrt  $M$ , with  $h = h_j^+$ ,  $h' = h_{j'}^+$  or  $h = h_j^-$ ,  $h' = h_{j'}^-$  and  $j < j' \leq p$ . For  $j < j'$ , i.e.,  $j \leq j' - 1$ , or  $2j \leq 2j' - 2$ , it follows

$$\beta_M + \frac{up_M - \beta_M}{2p}(2j) \leq \beta_M + \frac{up_M - \beta_M}{2p}(2j' - 2),$$

or  $up_{hM} \leq lo_{h'M}$ .

2.  $h$  and  $h'$  are negative wrt  $M$ : it is similar to the positive case.

3.  $h$  is negative and  $h'$  is positive wrt  $M$ : the result follows immediately from  $up_{hM} \leq \beta_M \leq lo_{h'M}$ .  $\square$

Now we show, that our membership modifiers can shift the membership degree of a given individual to the modified concept arbitrarily within  $(0, 1)$ . First, we prove the following lemma.

**Lemma 3.** Let  $M_q^* = \{M \mid length(M) = q\}$  be the set of all concept modifiers containing  $q$  hedges,  $(lo_M, up_M)$  be the interval for exponents of  $\delta M$  for every  $\delta$ . Then

$$\bigcup_{M \in M_q^*} [lo_M, up_M] = [lower^{q+1}, upper^{q+1}]$$

*Proof.* Let us denote  $M = hM'$ , with  $h \in H$  and  $M'$  contains  $q - 1$  hedges. By examining all four

cases (I), (II), (III) and (IV) we obtain:

$$\bigcup_{h \in H} [lo_{hM'}, up_{hM'}] = \begin{cases} [lower^{q+1}, lower^{q-1}] & \text{if } M' = \underline{M} \text{ is the} \\ & \text{greatest neg. modifier} \\ [upper^{q-1}, upper^{q+1}] & \text{if } M' = \overline{M} \text{ is the} \\ & \text{greatest pos. modifier} \\ [lo_{M'}, up_{M'}] & \text{otherwise} \end{cases}$$

Besides,

$$\bigcup_{M' \rightarrow \underline{M} \text{ and } M' \rightarrow \overline{M}} [lo_{M'}, up_{M'}] = [lower^{q-1}, upper^{q-1}]$$

Thus,

$$\begin{aligned} \bigcup_{M \in M_q^*} [lo_M, up_M] &= \\ \bigcup_{M' \in M_{q-1}^*} \left[ \bigcup_{h \in H} [lo_{hM'}, up_{hM'}] \right] &= \\ [lower^{q+1}, lower^{q-1}] \cup & \\ [lower^{q-1}, upper^{q-1}] \cup & \\ [upper^{q-1}, upper^{q+1}] & \\ = [lower^{q+1}, upper^{q+1}] & \end{aligned}$$

$\square$

**Theorem 2.** For  $x \in [0, +\infty)$  and every  $\epsilon > 0$  there exists a concept modifier  $M$  such that  $|\beta_M - x| < \epsilon$ , where  $\beta_M$  is the exponent for  $M$ .

*Proof.* For the given  $x$ , we can choose  $q$  such that  $x \in [lower^{q+1}, upper^{q+1}]$ . By Lemma 3 there exists  $M \in M_q^*$  such that  $x \in [lo_M, up_M]$ . So, we have  $|\beta_M - x| < up_M - lo_M$ . By examining the cases (I) and (III) it is easy to see that we find an  $h \in H$  such that  $x \in [lo_{hM}, up_{hM}]$  with  $up_{hM} - lo_{hM} \leq \frac{up_M - lo_M}{p}$ . This process continues until we obtain a  $\delta$  such that  $up_{\delta M} - lo_{\delta M} < \epsilon$  while  $x \in [lo_{\delta M}, up_{\delta M}]$ , and thus  $|\beta_{\delta M} - x| < \epsilon$ .  $\square$

It follows that, given an individual  $d$ ,  $m, n \in (0, 1)$  and  $\epsilon_m > 0$ , we can find a modifier  $M$  with an exponent  $\beta$ , such that

$$\langle d: A = n \rangle \models \langle d: MA = n^\beta \rangle$$

with  $|n^\beta - m| < \epsilon_m$  by applying the Theorem 2 with  $x = \frac{\log m}{\log n}$  and a suitably chosen  $\epsilon$ . Hence, our exponential membership modifiers generate a family of modified concepts from a primitive concept  $A$ ,

and change the membership degree of an individual as an element of those.

Furthermore, there is a subsumption relation between modified concepts.

**Theorem 3.** Let  $C = MA$  and  $D = M'A$  be two modified concepts from primitive concept  $A$ , and  $\beta, \beta'$  are the exponents of  $M, M'$ . Then,  $D$  subsumes  $C$  iff  $\beta \geq \beta'$ .

*Proof.*  $\Leftarrow$ :  $\beta \geq \beta'$  implies  $x^\beta \leq x^{\beta'}$  for all  $x \in [0, 1]$ , or  $(A^{\mathcal{I}}(d))^\beta \leq (A^{\mathcal{I}}(d))^{\beta'}$  for all individuals  $d$ . It follows  $(MA)^{\mathcal{I}}(d) \leq (M'A)^{\mathcal{I}}(d)$  for all  $d$ , or  $C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$ . Thus,  $D$  fuzzy subsumes  $C$ .

$\Rightarrow$ : Conversely, we need  $\beta \geq \beta'$  to hold so that  $(MA)^{\mathcal{I}}(d) \leq (M'A)^{\mathcal{I}}(d)$  or  $(A^{\mathcal{I}}(d))^\beta \leq (A^{\mathcal{I}}(d))^{\beta'}$  is true for all  $d$  and  $\mathcal{I}$ .  $\square$

Now we are ready to turn our attention to decision procedures.

## 4 Decision procedure

In this section we present a decision procedure for unsatisfiability in  $\mathcal{ALC}_{FH}$ . As usual, other problems can be reduced to an unsatisfiability problem, e.g. the satisfiability problem:

$$\begin{aligned} \Sigma \models \langle (w_1, w_2) : R \circ n \rangle \\ \text{iff } \Sigma \cup \langle (w_1, w_2) : R \bullet n \rangle \text{ is unsatisfiable,} \end{aligned}$$

$$\begin{aligned} \Sigma \models \langle w : C \circ n \rangle \\ \text{iff } \Sigma \cup \langle w : C \bullet n \rangle \text{ is unsatisfiable,} \end{aligned}$$

where  $\circ$  is one of  $>, \geq, \leq, <$ , and  $\bullet$  is the negated operator  $\leq, <, >, \geq$ , respectively. The assertion of type  $\langle \alpha = n \rangle$  is not explicitly considered here, since it can easily be reduced to assertions using  $\leq$  and  $\geq$ :

$$\begin{aligned} \Sigma \models \langle \alpha = n \rangle \quad \text{iff} \\ \Sigma \models \langle \alpha \leq n \rangle \quad \text{and} \quad \Sigma \models \langle \alpha \geq n \rangle, \end{aligned}$$

resp.

$$\begin{aligned} \Sigma, \langle \alpha = n \rangle \models \psi \quad \text{iff} \\ \Sigma, \langle \alpha \leq n \rangle, \langle \alpha \geq n \rangle \models \psi. \end{aligned}$$

In this section we restrict ourselves to pure assertional fuzzy knowledge bases  $\Sigma$ , i.e., knowledge bases without concept definitions and concept specializations. In section 7 we discuss how this can be extended.

Our unsatisfiability decision procedure closely follows [11], but introduces a new set of rules for the handling of concept modifiers. Starting from a set  $S$  of fuzzy constraints, we apply propagation rules to add ‘‘simpler’’ constraints preserving the satisfiability. This process continues until we either find some contradictory constraints (a *clash*), thus proving unsatisfiability, or if no more rules are applicable. In the latter case we can construct a model from the completed set of constraints. Due to the non-deterministic nature of some of the rules, unsatisfiability has to be proven for all possible choices in the application of the rules. Thus, in general backtracking will be necessary to prove unsatisfiability.

$S$  contains a *clash* iff it contains either one of the unsatisfiable fuzzy constraints  $\langle \perp \geq n \rangle$  where  $n > 0$ ,  $\langle \top \leq n \rangle$  where  $n < 1$ ,  $\langle \alpha < 0 \rangle$ ,  $\langle \alpha > 1 \rangle$ ,  $\langle \perp > n \rangle$ , or  $\langle \top < n \rangle$ , or  $S$  contains a conjugated pair of fuzzy constraints as in Table 3.

	$\langle \alpha < m \rangle$	$\langle \alpha \leq m \rangle$
$\langle \alpha \geq n \rangle$	$n \geq m$	$n > m$
$\langle \alpha > n \rangle$	$n \geq m$	$n \geq m$

Table 3: The conditions, under which a row-column pair of fuzzy constraints is conjugated.

Table 4 contains the rules for of calculus. We have augmented the rules from [11] by the rules  $(M_{>})$ ,  $(M_{\geq})$ ,  $(M_{\leq})$  and  $(M_{<})$  to handle the concept modifiers. All propagation rules have the form  $\Phi \rightarrow \Psi$  if  $\Gamma$ , where  $\Phi$  and  $\Psi$  are sets of fuzzy constraints and  $\Gamma$  is an optional condition defaulting to true. A rule can be applied to a set  $S$  of fuzzy constraints if the condition  $\Gamma$  holds wrt  $S$ ,  $\Phi \subseteq S$  and  $\Psi$  contains constraints not yet contained in  $S$ . As the result of the application the constraints in  $\Psi$  are added to  $S$ . An exception to this application rule are the non-deterministic rules  $(\sqcup_{\geq})$ ,  $(\sqcup_{>})$ ,  $(\sqcap_{\leq})$  and  $(\sqcap_{<})$ , where  $\Psi$  is of the form  $\Psi_1 \mid \Psi_2$ . They can be applied if  $\Gamma$  holds wrt  $S$ ,  $\Phi \subseteq S$  and neither  $\Psi_1$  nor  $\Psi_2$  is a subset of  $S$ . The result of their application is non-deterministically either the addition of  $\Psi_1$  or the addition of  $\Psi_2$  to  $S$ .

A *completion* of a finite set of constraints  $S$  is the result of the application of the rules in Table 4 starting with  $S$  until no more rules can be applied. The calculus has the *termination property*, i.e., the rules can only be applied finitely often to a finite set of constraints: Since the assertions the rules add to the set have smaller term sizes than the antecedents, and each rule can be applied only once (or finitely often in the case of  $(\forall_{\geq})$  and  $(\exists_{\leq})$ ) to the same antecedent, there can be no infinite chain of rule applications.

**Example 2.** Let us consider the following knowledge base  $\Sigma$  :

“If a customer is very interested in a property, he wants a product  $p$  that has that property” holds to a degree more than 0.9:

$$(1) \quad \langle p : \neg(\exists \text{attr}, \text{very int}) \sqcup \text{wants} \geq 0.9 \rangle$$

“The customer is interested in technics” to  $\geq 0.8$  :

$$(2) \quad \langle \text{tech} : \text{int} \geq 0.8 \rangle$$

“The product  $p$  has attribute technics” to  $\geq 0.7$  :

$$(3) \quad \langle (p, \text{tech}) : \text{attr} \geq 0.7 \rangle .$$

We want to prove that  $\Sigma$  entails that the customer wants product  $p$  to a degree of at least 0.9. That is, we have to prove the unsatisfiability of  $\Sigma$  together with the complement of the claim:

$$(4) \quad \langle p : \text{wants} < 0.9 \rangle$$

Application of rule  $(\sqcup_{\geq})$  is non-deterministic. We have to consider the case

$$(5) \quad \langle p : \text{wants} \geq 0.9 \rangle ,$$

which immediately yields a clash with (4). The other case yields

$$(5') \quad \langle p : \neg(\exists \text{attr}, \text{very int}) \geq 0.9 \rangle ,$$

$$(6') \quad \langle p : \exists \text{attr}, \text{very int} \leq 0.1 \rangle ,$$

Since (3) is conjugated to  $\langle (p, \text{tech}) : \text{attr} \leq 0.1 \rangle$ , application of  $(\exists_{\leq})$  to (6') yields:

$$(7') \quad \langle \text{tech} : \text{very int} \leq 0.1 \rangle$$

Application of  $(M_{\leq})$  with  $\text{exponent}(\text{very}) = 2$  gives

$$(8') \quad \langle \text{tech} : \text{int} \leq 0.316 \rangle ,$$

which clashes with (2). Thus, there is no clash-free completion of  $\Sigma \cup \{(4)\}$ . Hence, it is unsatisfiable, and consequently  $\Sigma \models \langle p : \text{wants} \geq 0.9 \rangle$ .

The discussed rules give us a decision procedure for  $ALC_{FH}$ , because we can construct a model for a set  $S$  of constraints from a clash-free completion of  $S$  or prove that there is no model for  $S$  if there is no clash-free completion.

**Theorem 4.** A finite set of fuzzy constraints  $S$  is satisfiable iff there exists a clash free completion of  $S$ .

$$(\neg_{\geq}) \quad \langle w : \neg C \geq n \rangle \rightarrow \langle w : C \leq 1 - n \rangle$$

$$(\neg_{\leq}) \quad \langle w : \neg C \leq n \rangle \rightarrow \langle w : C \geq 1 - n \rangle$$

$$(\sqcap_{\geq}) \quad \langle w : C \sqcap D \geq n \rangle \rightarrow \langle w : C \geq n \rangle, \langle w : D \geq n \rangle$$

$$(\sqcup_{\leq}) \quad \langle w : C \sqcup D \leq n \rangle \rightarrow \langle w : C \leq n \rangle, \langle w : D \leq n \rangle$$

$$(\sqcup_{\geq}) \quad \langle w : C \sqcup D \geq n \rangle \rightarrow \langle w : C \geq n \rangle \mid \langle w : D \geq n \rangle$$

$$(\sqcap_{\leq}) \quad \langle w : C \sqcap D \leq n \rangle \rightarrow \langle w : C \leq n \rangle \mid \langle w : D \leq n \rangle$$

$$(\forall_{\geq}) \quad \langle w_1 : \forall R. C \geq n \rangle, \psi \rightarrow \langle w_2 : C \geq n \rangle$$

if  $\psi$  is conjugated to  $\langle (w_1, w_2) : R \leq 1 - n \rangle$

$$(\exists_{\leq}) \quad \langle w_1 : \exists R. C \leq n \rangle, \psi \rightarrow \langle w_2 : C \leq n \rangle$$

if  $\psi$  is conjugated to  $\langle (w_1, w_2) : R \leq n \rangle$

$$(\exists_{\geq}) \quad \langle w : \exists R. C \geq n \rangle \rightarrow \langle (w, x) : R \geq n \rangle, \langle x : C \geq n \rangle$$

if  $x$  is a new variable and there is no  $w'$  such that both  $\langle (w, w') : R \geq n \rangle$  and  $\langle w' : C \geq n \rangle$  are already in the constraint set

$$(\forall_{\leq}) \quad \langle w : \forall R. C \leq n \rangle \rightarrow \langle (w, x) : R \geq 1 - n \rangle, \langle x : C \leq n \rangle$$

if  $x$  is a new variable and there is no  $w'$  such that both  $\langle (w, w') : R \geq 1 - n \rangle$  and  $\langle w' : C \leq n \rangle$  are already in the constraint set

$$(M_{\geq}) \quad \langle w : M(C) \geq n \rangle \rightarrow \langle w : C \geq n^{\frac{1}{\beta}} \rangle$$

if  $\beta = \text{exponent}(M)$

$$(M_{\leq}) \quad \langle w : M(C) \leq n \rangle \rightarrow \langle w : C \leq n^{\frac{1}{\beta}} \rangle$$

if  $\beta = \text{exponent}(M)$

Table 4: The rules of the decision procedure. In addition to the presented rules there are rules  $(\neg_{>})$ ,  $(\neg_{<})$ ,  $(\sqcap_{>})$  ...  $(\forall_{<})$ ,  $(M_{>})$ ,  $(M_{<})$  for the strict relations. These can easily be obtained from the rules above by replacing  $\geq$  by  $>$  and  $\leq$  by  $<$ .

*Proof.*  $\Rightarrow$ : By case analysis it is easily verified that the rules are sound, i.e., if we apply a rule to a satisfiable constraint-set  $S_1$ , the result  $S_2$  is satisfiable as well, and thus, clash-free. Let us just give two examples for the rules  $(\exists_{\leq})$  and  $(M_{\geq})$ .

$(\exists_{\leq})$ : Assume  $(\exists_{\leq})$  is applicable, i.e.,  $S_1$  contains  $\langle w_1 : \exists R.C \leq n \rangle$  and  $\langle (w_1, w_2) : R \geq n \rangle$  for some  $w_1, w_2$  and  $1 \geq m > n \geq 0$ .<sup>5</sup> Since  $S_1$  is satisfiable, there is an interpretation  $\mathcal{I}$  that satisfies both  $\langle w_1 : \exists R.C \leq n \rangle$  and  $\langle (w_1, w_2) : R \geq n \rangle$ . If  $\langle w_2 : C \leq m \rangle$  was not satisfied by  $\mathcal{I}$ , we would have  $\min\{R^{\mathcal{I}}(w_1^{\mathcal{I}}, w_2^{\mathcal{I}}), C^{\mathcal{I}}(w_2^{\mathcal{I}})\} = m > n$ , and thus  $(\exists R.C)^{\mathcal{I}} > n$  according to Table 2. This contradicts our assumption, i.e.,  $\langle w_2 : C \leq m \rangle$  is satisfied by  $\mathcal{I}$ , and thus  $S_2$ .

$(M_{\geq})$ : Assume  $(M_{\geq})$  is applicable, i.e.,  $S_1$  contains  $\langle w : MA \geq n \rangle$  for some  $W$ ,  $n$ , and the concept modifier  $M$  is applied to a primitive concept  $A$ . Since  $S_1$  is satisfiable, there is an interpretation  $\mathcal{I}$  that satisfies  $\langle w : MA \geq n \rangle$ , and thus  $(MA)^{\mathcal{I}}(w^{\mathcal{I}}) = (A)^{\mathcal{I}}(w^{\mathcal{I}})^{\beta} \geq n$ . Let  $\beta = \text{exponent}(M)$ . Since  $\beta > 0$  and  $n \geq 0$  we can conclude  $(A)^{\mathcal{I}}(w^{\mathcal{I}}) \geq n^{\frac{1}{\beta}}$ . Thus,  $S_2$  is satisfied by  $\mathcal{I}$  as well.

$\Leftarrow$ : Assume  $S'$  is a clash-free completion of  $S$ . We will now construct a model for the fuzzy constraints of  $S'$  that only contain primitive concepts or roles, and prove that it is a model of  $S'$ , and thus  $S$ , as well.

Consider the interpretation  $\mathcal{I}$  such that the domain  $\Delta^{\mathcal{I}}$  is the set of objects appearing in  $S'$ ,  $w^{\mathcal{I}} = w$  for all  $w \in \Delta^{\mathcal{I}}$ , and that maps the primitive concepts and the roles to the median of the lowest upper bound and the greatest lower bound given in  $S'$ , inclusive the implicit constraints 0 and 1:

$$A^{\mathcal{I}}(w^{\mathcal{I}}) = \frac{1}{2} \max \left( \begin{array}{l} \{n \mid \langle w : A > n \rangle \in S'\} \cup \\ \{n \mid \langle w : A \geq n \rangle \in S'\} \cup \{0\} \end{array} \right) + \frac{1}{2} \min \left( \begin{array}{l} \{n \mid \langle w : A < n \rangle \in S'\} \cup \\ \{n \mid \langle w : A \leq n \rangle \in S'\} \cup \{1\} \end{array} \right)$$

$$R^{\mathcal{I}}(w_1^{\mathcal{I}}, w_2^{\mathcal{I}}) = \frac{1}{2} \max \left( \begin{array}{l} \{n \mid \langle (w_1, w_2) : R > n \rangle \in S'\} \cup \\ \{n \mid \langle (w_1, w_2) : R \geq n \rangle \in S'\} \cup \\ \{0\} \end{array} \right) + \frac{1}{2} \min \left( \begin{array}{l} \{n \mid \langle (w_1, w_2) : R < n \rangle \in S'\} \cup \\ \{n \mid \langle (w_1, w_2) : R \leq n \rangle \in S'\} \cup \\ \{1\} \end{array} \right)$$

<sup>5</sup>We skip the similar other case  $\langle w_2 : C > m \rangle$  with  $m \geq n$  for the clash, here.

It can easily be verified that this interpretation satisfies all constraints for primitive concepts and roles given in  $S'$  if  $S'$  is clash-free. The satisfaction of the other fuzzy constraints in  $S'$  are shown by induction on the term structure of the  $ALC_{FH}$ -formula in the constraints. Again, we limit ourselves to the presentation of two cases for space reasons.

Case  $\langle w : \forall R.C < n \rangle$ : Since  $S'$  is complete, there is a  $w'$  such that  $\langle (w, x) : R > 1 - n \rangle$  and  $\langle x : C < n \rangle$  are in  $S'$  and are satisfied by  $\mathcal{I}$  by induction assumption.<sup>6</sup> Thus, there is a  $w' \in \Delta^{\mathcal{I}}$  such that  $1 - R^{\mathcal{I}}(w^{\mathcal{I}}, w'^{\mathcal{I}}) < n$  and  $C^{\mathcal{I}}(w'^{\mathcal{I}}) < n$ , and hence  $(\forall R.C)^{\mathcal{I}} < n$ .

Case  $\langle w : MA > n \rangle$ : Since  $S'$  is complete,  $\langle w : A > n^{\frac{1}{\beta}} \rangle$  with  $\beta = \text{exponent}(M)$  is in  $S'$  and is satisfied by  $\mathcal{I}$  by induction assumption. Thus,  $A^{\mathcal{I}}(w^{\mathcal{I}}) > n^{\frac{1}{\beta}}$ , and since  $\beta > 0$  we have that  $(MA)^{\mathcal{I}}(w^{\mathcal{I}}) > n$ .  $\square$

## 5 Truth Bounds and Expressivity

In practice it is often important to determine bounds for the truth values at which fuzzy constraints are entailed or can be satisfied. For instance, for a knowledge base  $\Sigma$  the lowest value of  $n$  (called the *lower truth bound*) such that  $\Sigma \models \langle k : \text{technical} \geq n \rangle$  might determine how much the customer  $k$  is interested in technical aspects. [11] proves for Fuzzy- $ALC$  that  $n$  is one of the values occurring in  $\Sigma$ , their complements, or one of  $\{0, 0.5, 1\}$ . The following example shows that this is not true for our language. Thus, the binary search in the set suggested in [11] does not suffice to calculate the truth values in  $ALC_{FH}$ . This also shows that our language is strictly more expressive than Fuzzy- $ALC$ .

**Example 3.** Consider the satisfiability problem

$$(9) \quad \emptyset \models \langle \text{technical} \sqcap \neg \text{very technical} \geq n \rangle.$$

As the reader can easily verify, this is satisfiable whenever  $n$  is less than  $1 - x$ , where  $x$  is the solution for  $1 - x^{\beta} = x$  and  $\beta = \text{exponent}(\text{very})$ . If we choose the hedges as in Example 1, we have  $\beta = 2$ , and thus (9) is satisfiable iff  $n \geq 0.618\dots$

However, one can easily approximate the truth bounds to any given precision by a binary search in the interval  $[0, 1]$ . E.g., starting from  $[0, 1]$ , an interval  $[l, u]$  for the lower truth bound for  $n$  in  $\Sigma \models \langle k : C \geq n \rangle$  can successively be

<sup>6</sup> $w'$  is either the new variable introduced by the application of  $(\forall_{<})$ , or the  $w'$  that prevented the execution of  $(\forall_{<})$ .

halved to  $[\frac{u+l}{2}, u]$  or  $[l, \frac{u+l}{2}]$  by checking whether  $\Sigma \models \langle k : C \geq \frac{u+l}{2} \rangle$  holds. It seems likely that the application of constraint propagation techniques can improve this process very much.

## 6 Complexity

Because the language Fuzzy-*ALC* defined in [11] is a subset of  $\mathcal{ALC}_{FH}$  and we use a similar decision procedure, the PSPACE-hardness of deciding entailment for Fuzzy-*ALC* carries over to our language. Another result that carries over is, that the rules  $(\exists_{\geq})$ ,  $(\exists_{>})$ ,  $(\forall_{\leq})$  and  $(\forall_{<})$  for the quantifiers  $\exists$  and  $\forall$  can lead to exponential space requirements in a proof. [11] alleviates this problem through the introduction of so called *trace rules* for the quantifiers which reduce the number of variables that need to be introduced in each completion, but modify the decision procedure by introducing another kind of non-determinism needed to still cover all possibilities. We believe this can be done in our case the same way. We try to give the reader a flavor of this by the following small example, but refer the reader to [11] for a more detailed discussion.

**Example 4.** Consider a set of constraints

$$S = \{ \langle e : (\exists r.c) \geq 0.5 \rangle, \langle e : (\exists r.d) \geq 0.5 \rangle \} \cup T ,$$

where no rule other than  $(\exists_{\geq})$ ,  $(\exists_{>})$ ,  $(\forall_{\leq})$  or  $(\forall_{<})$  is applicable. The rule  $(\exists_{\geq})$  would usually introduce two new variables  $x$  and  $y$  yielding a set

$$(10) \quad S' = S \cup \{ \langle \langle (w, x) : r \geq n \rangle, \langle x : c \geq n \rangle, \langle (w, y) : r \geq n \rangle, \langle y : d \geq n \rangle \rangle \} .$$

As it turns out there is no need to have both new variables at the same time in the set, but it is possible to check both of them independently. So we introduce a trace rule corresponding to  $(\exists_{\geq})$ :

$$(T\exists_{\geq}) \quad \langle w : \exists R.C \geq n \rangle \rightarrow \langle \langle (w, x) : R \geq n \rangle, \langle x : C \geq n \rangle \rangle$$

if  $x$  is a new variable  
and no  $\langle (w, w') : R \geq n \rangle$

is already in the constraint set.<sup>7</sup>

The rule  $(T\exists_{\geq})$  is applicable to  $S$  in two ways that yielding either

$$S^{(1)} = S \cup \{ \langle \langle (w, x) : r \geq n \rangle, \langle x : c \geq n \rangle \rangle \}$$

<sup>7</sup>The reader may note, that at most one new variable is introduced per role and  $W$ . This is not the case in  $(\exists_{\geq})$ .

or

$$S^{(2)} = S \cup \{ \langle \langle (w, y) : r \geq n \rangle, \langle y : d \geq n \rangle \rangle \} .$$

In our decision algorithm it was sufficient to check whether  $S$  has at least one clash-free completion to conclude that it is satisfiable. However, when using trace rules it is necessary to check that all ways of applying the trace rule, namely  $S^{(1)}$  and  $S^{(2)}$ , have a clash-free completion at the same time. On the other hand each considered completion uses only polynomial space.

## 7 Considering terminologies

So far, we have been concerned only with pure assertional knowledge bases, leaving out concept definitions and specializations. Since  $\mathcal{ALC}_{FH}$  is a conservative extension of Fuzzy-*ALC*, the handling of terminologies is the same as the approach used there in [11]—we use knowledge base expansion as devised in [10]. In short, for each fuzzy knowledge base  $\Sigma = \Sigma_A \cup \Sigma_T$ , where  $\Sigma_A$  contains only fuzzy assertions and  $\Sigma_T$  contains only concept definitions and concept specializations, we find a fuzzy knowledge base  $\Sigma'_A$  that contains only assertions, such that  $\Sigma'_A$  is satisfiable iff  $\Sigma$  is satisfiable. Roughly, the expansion process works as follows:

1. We obtain  $\Sigma'_T$  by replacing all concept specializations  $A \leq C$  by concept definitions  $A := C \sqcap A^*$  with a new primitive concept  $A^*$  per concept  $A$ .
2.  $\Sigma'_A$  is created from  $\Sigma_A$  by replacing every primitive concept  $C$  that has a definition  $C := D$  in  $\Sigma'_T$  iteratively  $\Sigma_A$  by its definition  $D$ . This process repeats until there are no more defined concepts in  $\Sigma'_A$ .

Of course, this process requires that the concept definitions in  $\Sigma_T$  are not cyclic and there is only one concept definition per concept. For a more extensive discussion we refer to [11].

## 8 Subsumption

Like in Fuzzy-*ALC*, a subsumption problem  $C \preceq_{\Sigma_T} D$  can be reduced to an equivalent subsumption problem  $C' \preceq_{\emptyset} D'$  with an empty terminology by an expansion process similar to the process discussed in the previous section. Unfortunately, the subsumption decision algorithm of Fuzzy-*ALC* in [11] does not work in our logic due

to the richer structure of concepts. In Fuzzy-*ALC*,  $C \preceq D$  holds iff  $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$  holds for  $m = n_1, n_2$  for some arbitrary values  $0 < n_1 \leq \frac{1}{2}$  and  $\frac{1}{2} < n_2 \leq 1$ . However, in *ALC<sub>FH</sub>* this is not true, as the following example shows. Consider the subsumption problem  $A \preceq (A \sqcap (\text{mol mol } B \sqcup \neg \text{very very } B))$ . Obviously, this is not valid. However, it is easy to check that  $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$  holds for, say,  $n_1 = 0.4$  and  $n_2 = 0.6$ , assuming  $\text{exponent}(\text{mol mol}) = \frac{1}{4}$  and  $\text{exponent}(\text{very very}) = 4$ .<sup>8</sup> We are currently looking for another way to solve the subsumption problem.

## 9 Discussion

Description logics are a widely accepted formalism for representing and reasoning about a terminology based on crisp concepts and relations. Fuzzy logic is a widely accepted formalism for representing and reasoning about non-crisp concepts and relations. Surprisingly, there have been only few attempts to combine the advantages of both within one system that we are aware of, viz. [12] and [11].

In this paper we have presented the fuzzy description logic with hedges *ALC<sub>FH</sub>*. It is a conservative extension of Fuzzy-*ALC* defined by Umberto Straccia in [11]. In [11] the classical description logic *ALC* is extended by interpreting concepts and relations as fuzzy sets, the correspondence of the fuzzy semantics and the semantics of the equivalent “crisp” *ALC* is discussed and decision procedures for the entailment and subsumption problem have been defined. We extend the fuzzy description logic given there by concept manipulators whose semantics is based on hedge algebras.

Christopher B. Tresp and Ralf Molitor [12] extend *ALC* with membership manipulators, i.e., functions that map the set  $[0, 1]$  of fuzzy truth values to  $[0, 1]$ . These can be applied to the fuzzy sets representing concepts in order to construct new concepts. The membership manipulators are restricted to triangular functions on  $[0, 1]$ . Tresp and Molitor also define a constraint propagation based method for computing the degree of subsumption between two concepts. However, their consequence relation is counter-intuitive to us in some cases. In [12] a concept  $D$  is subsumed by a concept  $C$  by the minimum degree of  $D$  being satisfied in interpretations that map  $C$  to 1. Consequently, they can draw arbitrary conclusions from a premise which is never

<sup>8</sup>This is since  $(\text{mol mol } B \sqcup \neg \text{very very } B)^{\mathcal{I}} \in [0.877, 1]$  for all  $B^{\mathcal{I}}$ .

satisfied to degree 1, but, say, only to 0.6. Such a concept is discussed in Example 3. In *ALC<sub>FH</sub>* membership manipulators are hedges and the subsumption of modified concepts is based—as we believe quite intuitively—on the “less than or equal” relation of the corresponding  $\beta$ -values as shown in Theorem 3.

We hope that this paper is the starting point of a fruitful combination of description logic, fuzzy logic and hedge algebras based on the idea of manipulating concepts by hedges. Indeed, many interesting problems are yet to be solved. In [8] we admitted concept modifiers only on primitive concepts, whereas we now admit modifiers on complex concepts as well. A problem with the approach chosen here is that there are some cases that are likely to confuse the user. Consider the term *very (mol int)*. Isn’t it natural to expect, that this term has the same meaning as *very mol int*? However, this is not what happens:

$$(\text{very}(\text{mol int}))^{\mathcal{I}} = \eta_{\text{very}}(\eta_{\text{mol}}(\text{int}^{\mathcal{I}})) ,$$

which is in general different than

$$(\text{very mol int})^{\mathcal{I}} = \eta_{\text{very mol}}(\text{int}^{\mathcal{I}}) ,$$

like, for example, when  $\text{exponent}(\text{very}) = 2$ ,  $\text{exponent}(\text{mol}) = 0.5$  and  $\text{exponent}(\text{very mol}) = 0.75$  as in example 1. We are looking for a solution that captures the intuition that the discussed terms have the same meaning.

In this paper we have only discussed the unsatisfiability problem. We believe that trace rules introduced in [11] for Fuzzy-*ALC* can be extended to deal with *ALC<sub>FH</sub>*, but this remains to be rigorously shown. Whether there is a solution to the subsumption problem as simple as in [11] is topic of further research—as discussed, Straccia’s method does not work unmodified.

Finding a good real-world application is vital for the success of the proposed method. We believe that the field of intelligent user-adaptive e-commerce systems provides such examples. We are envisioning the use of the services provided by Fuzzy-*ALC* within a fuzzy rule engine in a web shop system [3]. In such a shop the products are provided with fuzzy attributes describing properties and categorial memberships. The behavior of a user is used to mechanically generate corresponding attributes indicating the users interests. Given those attributes as concepts, the shop operator can define user stereotypes in Fuzzy-*ALC*, such that the membership of a user to one of those stereotypes can be decided with the help of the satisfiability algorithm in Fuzzy-*ALC*. When the user is shown a new web

pages describing products, the presentation can be adapted in various ways, such as giving personalized recommendations based on the stereotype memberships, or changing the layout of the web page to emphasize product information that the system supposes to be of particular interest to the user. Unfortunately, these experiments are in their initial stage only and we have no detailed results yet.

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