The Subsumption Problem of the Fuzzy Description Logic $\mathcal{A}LC_{FH}$

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Abstract

We present the fuzzy description logic $\mathcal{A}LC_{FH}$ where primitive concepts are modified by means of hedges taken from hedge algebras. $\mathcal{A}LC_{FH}$ is strictly more expressive than Fuzzy-$\mathcal{A}LC$ defined in [9]. We show that given a linearly ordered set of hedges primitive concepts can be modified to any desired degree by prefixing them with appropriate chains of hedges. Furthermore, we define a decision procedure for the unsatisfiability problem in $\mathcal{A}LC_{FH}$ and for the subsumption problem in a fragment of $\mathcal{A}LC_{FH}$.

Key words: fuzzy logic, hedge algebras, description logic.

1 Introduction

In many areas of computer science and artificial intelligence logic has been accepted as the mathematical foundation for knowledge representation and reasoning. Beyond that it has been recognized that the logic itself can be used as computation unifying in an almost ideal way declarative and operational semantics. Among the many applications description logics have been particularly successful (see e.g. [1, 2]). The main task of a description logic is to fix a terminology by describing concepts and their relations, and to provide (decidable) key services for reasoning wrt this terminology like, for example, subsumption and unsatisfiability testing.

In all applications of description logics that we are aware of, except [9], [10] and [7], concepts are crisp unary relations, i.e., an object may or may not be an element of a particular concept. On the other hand, in many real-world applications like, for example, intelligent e-commerce information is often vague and imprecise. We may observe that a customer is interested in technical aspects, whereas he or she is not interested in design issues. Fuzzy set theory introduced by Lotfi A. Zadeh (see e.g. [12]) provides an ability to denote non-crisp concepts, i.e., an object may belong to a certain degree (typically a real number from the interval $[0, 1]$) to a particular relation.

Humans typically use linguistic adverbs like "very", "more or less" etc. to distinguish, for example, between a customer who is interested in technical details and one who is very interested in these details. In [11] Zadeh introduces so-called linguistic hedges modifying the shape of a fuzzy set by transforming it into another. For instance, he introduced the operator *CON* (for "contraction") that maps the membership function μ_A of a fuzzy set A to a membership function $\mu_{very\,A}$ that has high degrees of membership only for those elements of the domain, that belong "very much" to A. Technically, Zadeh achieves this by simply raising the degree of membership to the β-th power, where $\beta > 1$ is a constant which can be fixed by the application. For example, $\mu_{very\,A}(u) = \mu_A(u)^2$. This technical solution calls for the introduction of an opposite operator *DIL* (for dilation) that maps the degree of membership to the β'-th power, where $0 < \beta' < 1$, thereby strengthening the degree of membership

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of elements with small degree. One should note that these hedges can be concatenated. For example, if we want to consider elements that belong to A "very very much", then μ_A can be transformed into its β^2-th power by applying the operator *CON* twice: *CON CON* A.

Because the gap between A and *CON* A in this setting is quite large, it seems a good idea to refine this setting. In many human languages there is almost a continuum of phrases like "more or less", "much less", "possibly rather" and so forth expressing different levels of emphasis. The so called *hedge algebras* introduced in [5, 4] give an algebraic structure that formally defines such hedges and structures their relationships. They have been applied to fuzzy logic in various ways (see e.g. [3]).

In this paper we apply linear ordered hedge algebras as concept modifiers in the framework of a fuzzy description logic. The paper is organized as follows: In the following Section 2 we recall basic notions from fuzzy description logics and the structure of hedges. In particular, we define an extended fuzzy description logic with hedges $\mathcal{A}LC_{FH}$. Concept modifiers are formally introduced in Section 3, and we describe an algorithm that maps each concept modifier to an operator on membership functions. A decision procedure for the unsatisfiability problem of $\mathcal{A}LC_{FH}$ is presented in Section 4. Section 5 introduces a decision procedure that determines subsumptions for a fragment of $\mathcal{A}LC_{FH}$. Finally, we discuss our findings in Section 6.

2 The Fuzzy Description Logic $\mathcal{A}LC_{FH}$

Concepts are expressions that collect properties of a set of individuals, described–among others–by means of roles. From a first order logical viewpoint, concepts can be seen as unary predicates, whereas roles are interpreted as binary predicates. A concept (denoted throughout the paper by C or D) is either a primitive concept, denoted by A, a modified concept MC (concept modifiers M are discussed later), or one of $\top$, $\bot$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$ or $\exists R.C$, where R denotes a role. For the fuzzy extension, a concept C, rather than being interpreted as a classical set, will be represented as a fuzzy set and, thus, concepts become imprecise. Consequently, a statement like "a is C", where a is an individual, will have a truth-value in $[0,1]$ denoting the degree of the membership of a in the fuzzy set C.

The semantics is based on the notion of an interpretation. An interpretation is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of an non-empty set $\Delta^{\mathcal{I}}$ (called the domain) and an interpretation function $\cdot^{\mathcal{I}}$ mapping individuals to elements of $\Delta^{\mathcal{I}}$, concepts C to a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$, and roles R to a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. Therefore, if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$, then $C^{\mathcal{I}}(d)$ is the degree of d being an element of the fuzzy concept C under $\mathcal{I}$. Roles are interpreted in the same way. The interpretation for complex concepts is defined in Table 1.[1]

$$
\begin{aligned}
A^{\mathcal{I}} &: \Delta^{\mathcal{I}} \to [0,1] \\
R^{\mathcal{I}} &: \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1] \\
\top^{\mathcal{I}}(d) &= 1 \text{ for all } d \in \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}}(d) &= 0 \text{ for all } d \in \Delta^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}}(d) &= \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) &= \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) &= 1 - C^{\mathcal{I}}(d) \\
(M(C))^{\mathcal{I}}(d) &= \eta_M(C^{\mathcal{I}}(d)) \\
(\forall R.C)^{\mathcal{I}}(d) &= \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{1 - R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) &= \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d')\}\}
\end{aligned}
$$

Table 1: Interpretations of $\mathcal{A}LC_{FH}$. η_M is a membership modifier discussed later.

Two concepts C and D are said to be *equivalent* (denoted by $C \cong D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$. For instance, $\top \cong \neg\bot$, $(C \sqcap D) \cong \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong \neg(\exists R.\neg C)$.

A *fuzzy assertion* (denoted by ψ) is one of $\langle \alpha > n \rangle$, $\langle \alpha \geq n \rangle$, $\langle \alpha \leq n \rangle$, $\langle \alpha < n \rangle$ and $\langle \alpha = n \rangle$ where α is an expression of type $a : C$ ("a is in C"), $(a,b) : R$ ("(a,b) is in R") and $n \in [0,1]$. From a semantical viewpoint, a fuzzy assertion $\langle \alpha \geq n \rangle$ constrains the fuzzy truth-value [2] of α to be greater or equal to n. For-

[1] The similarity of the definitions of many of the operators defined in Table 1 with fuzzy set operations [8] is intended.

[2] This fuzzy truth value corresponds to the membership value of a in C when $\alpha = a : C$, or of (a,b) in R

mally, an interpretation $\mathcal{I}$ satisfies a fuzzy assertion $\langle a : C \geq n \rangle$ (resp. $\langle (a,b) : R \geq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). The operators $>, =, <$ and $\leq$ are interpreted in the same way. Two fuzzy assertion ψ_1 and ψ_2 are said to be *equivalent* (denoted by $\psi_1 \cong \psi_2$) iff they are satisfied by the same set of interpretations.

A *fuzzy terminological axiom* is either a *fuzzy concept specialization* of the form $A :\leq C$ or a *fuzzy concept definition* of the form $A := C$. A specialization denotes the "more specific than"-relation between concepts, and a definition denotes the equivalence of concepts. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy concept specialization $A :\leq C$ iff for all $d \in \Delta^{\mathcal{I}}$ we find $A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d)$, whereas $\mathcal{I}$ satisfies a fuzzy concept definition $A := C$ iff for all $d \in \Delta^{\mathcal{I}}$ we find $A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d)$.

A *fuzzy knowledge base* Σ is the union of a finite set Σ_A of fuzzy assertions and a finite set Σ_T of fuzzy terminological axioms. An interpretation $\mathcal{I}$ *satisfies* (or is a *model* of) a fuzzy knowledge base Σ iff $\mathcal{I}$ satisfies each element of Σ. A fuzzy knowledge base *entails* a fuzzy assertion ψ (denoted by $\Sigma \models \psi$) iff every model of Σ also satisfies ψ. Furthermore, let C and D be two concepts, D *fuzzy subsumes* C with respect to Σ_T (denoted by $C \preceq_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of Σ and for all $d \in \Delta^{\mathcal{I}}$ we find $C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$.

The problem of determining whether $\Sigma \models \psi$ is called *entailment problem*, whether $C \preceq_{\Sigma_T} D$ is called *fuzzy subsumption problem*, and whether Σ is satisfiable is called *satisfiability problem*.

3 Concept Modifiers

Let us consider the set $H = \{h_1, h_2, ..., h_p\}$ of hedges. Each of these hedges either positive (i.e. increases the meaning) or negative (i.e. decreases the meaning) wrt all primitive concepts and all other hedges including itself.

In $\mathcal{ALC}_{FH}$ we introduce concept modifiers M in form of a chain of hedges $M = k_q k_{q-1} \ldots k_1$ with $k_i \in H$, for all $i = \overline{1,q}$. The mapping $\eta_M : [0,1] \rightarrow [0,1]$ gives each hedge chain a meaning within the interpretation defined in Table 1. We will follow the idea of Zadeh to use power functions for this purpose.

Definition 1. A membership modifier is an exponential function $\eta : [0,1] \rightarrow [0,1]$ with $\eta(x) = x^{\beta}$, where $\beta > 0$.

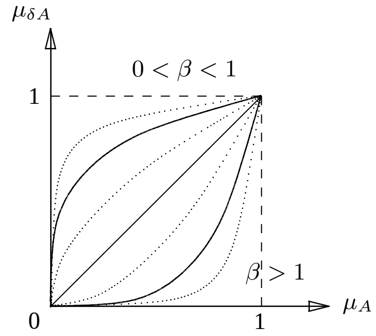

Figure 1: The membership modifier of δA.

So we have to calculate an exponent β for each sequence of hedges. Our method to calculate β is based on following ideas.

- We extend the idea behind Zadeh's operators *CON* and *DIL* discussed in the introduction in that we keep the idea of taking a power function with exponents $\beta > 1$ for positive and $0 < \beta < 1$ for negative hedges, but vary the exponents depending on the hedge.

- Membership modifiers should preserve the independence property of hedges. I.e., for all $h, h', k, k' \in H$ we should find that if $h\delta < k\delta$ then $h'h\delta < k'k\delta$, where δ is a chain of hedges. In other words, h' and k' can not change the semantic ordering between hx and kx. As we will show, this property carries over to the exponent, and thus, the modified concepts. Note that this is different from Zadeh's proposal of *CON* and *DIL* : they are commutative: *CON DIL* $A =$ *DIL CON* A, whereas we have *very rather* $<$ *rather very* if *rather* $<$ *very* .

- The set $H^* = \{\delta \mid \delta$ is a chain of hedges $\}$ of all membership modifiers should be mapped dense into the interval $(0, \infty)$. I.e. we should be able to approximate each value in $(0, \infty)$ to any required precision by an $exponent(\delta)$ such that $\delta \in H^*$. In case of β approaches 0, the correlative membership

when $\alpha = (a,b) : R$.

modifier may change membership degrees of all individuals to 1, and in inverse case, if β comes close to $+\infty$, then all membership degrees approximate 0 (see Figure 1).

Based on the above ideas, an algorithm to calculate an exponent β for each sequence of hedges is given in [6].

Example 1. Consider $H = \{\textit{very}, \textit{mol}^3\}$. (*mol* is an abbreviation for "more or less".) We obtain:
exponent(very)=2, exponent(mol)=0.5,
exponent(very very)=4, exponent(mol very)=1.5,
exponent(very mol)=0.75, exponent(mol mol)=0.25.

Now we are ready to turn our attention to decision procedures.

4 The Unsatisfiability Problem

In this section we present a decision procedure for unsatisfiability in $\mathcal{ALC}_{FH}$. As usual, other problems can be reduced to an unsatisfiability problem, e.g. the entailment problem: $\Sigma \models \langle w : C \circ n \rangle$ iff $\Sigma \cup \langle w : C \bullet n \rangle$ is unsatisfiable, where $\circ$ is one of $>, \geq, \leq, <$, and $\bullet$ is the negated operator $\leq, <, >, \geq$, respectively. We restrict ourselves to pure assertional fuzzy knowledge bases Σ, i.e., knowledge bases without concept definitions and concept specializations. For the extension we refer to [9].

Our unsatisfiability decision procedure closely follows [9], but introduces a new set of rules for the handling of concept modifiers. Starting from a set S of fuzzy constraints, we apply propagation rules to add "simpler" constraints preserving the satisfiability. This process continues until we either find some contradictory constraints (a *clash*), thus proving unsatisfiability, or if no more rules are applicable.

S contains a *clash* iff it contains either one of the unsatisfiable fuzzy constraints $\langle \bot \geq n \rangle$ where $n > 0$, $\langle \top \leq n \rangle$ where $n < 1$, $\langle \alpha < 0 \rangle$, $\langle \alpha > 1 \rangle$, $\langle \bot > n \rangle$, or $\langle \top < n \rangle$, or S contains a conjugated pair of fuzzy constraints as:
$\langle \alpha \geq n \rangle$ and $\langle \alpha < m \rangle$ where $n \geq m$,
$\langle \alpha > n \rangle$ and $\langle \alpha \leq m \rangle$ where $n \geq m$,
$\langle \alpha > n \rangle$ and $\langle \alpha < m \rangle$ where $n \geq m$,
$\langle \alpha \geq n \rangle$ and $\langle \alpha \leq m \rangle$ where $n > m$.

We have augmented the rules from [9] by the rules $(M_>)$, $(M_\geq)$, $(M_\leq)$ and $(M_<)$ to handle the concept modifiers.

$$(M_>) \quad \langle w : M(C) > n \rangle \quad \rightarrow \langle w : C > n^{\frac{1}{\beta}} \rangle \quad \text{if} \quad \beta = \textit{exponent}(M)$$

$$(M_\geq) \quad \langle w : M(C) \geq n \rangle \quad \rightarrow \langle w : C \geq n^{\frac{1}{\beta}} \rangle \quad \text{if} \quad \beta = \textit{exponent}(M)$$

$$(M_\leq) \quad \langle w : M(C) \leq n \rangle \quad \rightarrow \langle w : C \leq n^{\frac{1}{\beta}} \rangle \quad \text{if} \quad \beta = \textit{exponent}(M)$$

$$(M_<) \quad \langle w : M(C) < n \rangle \quad \rightarrow \langle w : C < n^{\frac{1}{\beta}} \rangle \quad \text{if} \quad \beta = \textit{exponent}(M)$$

The propagation rules have the form $\Phi \rightarrow \Psi$ if Γ, where Φ and Ψ are sets of fuzzy constraints and Γ is an optional condition defaulting to true. A rule can be applied to a set S of fuzzy constraints if the condition Γ holds wrt S, $\Phi \subseteq S$ and Ψ contains constraints not yet contained in S. As the result of the application the constraints in Ψ are added to S.

A *completion* of a finite set of constraints S is the result of the application of the propagation rules starting with S until no more rules can be applied. The calculus has the *termination property*, i.e., the rules can only be applied finitely often to a finite set of constraints: Since the assertions the rules add to the set have smaller term sizes than the antecedents, and each rule can be applied only once (or finitely often in the case of $(\forall_\geq)$ and $(\exists_\leq)$) to the same antecedent, there can be no infinite chain of rule applications.

5 The Subsumption Problem

As the main contribution of this paper, we describe in this section an algorithm that solves subsumption problems in the fragment of $\mathcal{ALC}_{FH}$, where modifiers are only applied to primitive concepts. How to extend this to larger fragments remains an open problem.

Like in Fuzzy-$\mathcal{ALC}$, a subsumption problem $C \preceq_{\Sigma_T} D$ can be reduced to an equivalent subsumption problem $C' \preceq_\emptyset D'$ with an empty terminology by an expansion process similar to the process discussed in the previous section. As a next step, we reduce the subsumption problem to a class of satisfiability problems.

Proposition 1. Let C and D be two concepts of $\mathcal{ALC}_{FH}$. $C \preceq_\emptyset D$ iff for all $m > 0$ we have that $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$, where a is a new individual.

We omit the proof of this proposition for space reasons, since the proof of the corresponding proposition in [9] for Fuzzy-$\mathcal{ALC}$ carries over word by word to $\mathcal{ALC}_{FH}$.

But how can we check $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$ for the infinitely many values of $m \in (0, 1]$? Unfortunately, the approach for Fuzzy-$\mathcal{ALC}$ in [9] does not work in our logic due to the richer structure of concepts. In Fuzzy-$\mathcal{ALC}$, it is sufficient to check this for only two values $m = m_1$ and $m = m_2$ chosen arbitrarily such that $0 < m_1 \leq 1/2$ and $1/2 < m_2 \leq 1$. If $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$ holds for both of these two values, it will hold for all other values as well. However, in $\mathcal{ALC}_{FH}$ this is not true as the following example shows. Consider the subsumption problem $A \preceq (A \sqcap (\textit{mol mol}\, B \sqcup \neg \textit{very very}\, B))$. Obviously, this is not valid. On the other hand it is easy to check that $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$ holds for, say, $m_1 = 0.4$ and $m_2 = 0.6$, assuming $\textit{exponent}(\textit{mol mol}) = \frac{1}{4}$ and $\textit{exponent}(\textit{very very}) = 4$.[4] So we have to find another way to tackle such problems.

Let us restate the problem. To determine whether $C \preceq_{\emptyset} D$ we have to verify that $\langle a : C \geq m \rangle \models \langle a : D \geq m \rangle$ holds for all $m \in (0, 1]$. This holds, iff $\{\langle a : C \geq m \rangle, \langle a : D < m \rangle\}$ has no clash free completion for any value $m \in (0, 1]$. We will do this by trying to find all completions $\widetilde{\mathcal{S}}$ of $\mathcal{S} = \{\langle a : C \geq s \rangle, \langle a : D < s \rangle\}$ while keeping s as a variable symbol in S, and only afterwards check for which values of s we will find clashes. We call this process *symbolic completion*. A fuzzy assertion $\langle x : C \circ t \rangle$ or $\langle (w_1, w_2) : R \circ t \rangle$ where $\circ$ is one of $>, \geq, \leq, <$ and t is a term–instead of a number–is called a *symbolic fuzzy assertion*. t is called the right hand side of the fuzzy assertion. We can map symbolic completions back to completions by replacing s with numbers from $[0, 1]$.

For reasons that will become fully apparent later, we have to perform this process twice: once for values of m in $(0, 0.5)$, and once for values of m in $(0.5, 1.0)$. We use two different symbolic values s_l and s_u to represent m in these intervals. Together with this modification, the restriction of modifiers to primitive concepts allows us to perform the conjugacy checks in rules $(\forall_{\geq})$ $(\forall_{>})$, $(\exists_{<})$ and $(\forall_{\leq})$ just from the restriction to those intervals.

Definition 2. A *symbolic rule application* is the application of the propagation rules while observing the conjugacy conditions in Table 2 and $1 - (1 - s_l) = s_l$, $1 - (1 - s_u) = s_u$

A *symbolic completion* of a set of symbolic fuzzy assertions S is the result of symbolic rule applications starting with S until no more rules can be applied. When applying the rules $(M_<)$, $(M_{\leq})$, $(M_{\geq})$ or $(M_>)$, the term $n^{\frac{1}{\beta}}$ is kept as a term. E.g., when $M_<$ is applied to $\langle a : C \geq (1 - s_l) \rangle$ the fuzzy assertion $\langle a : C \geq (1 - s_l)^{\frac{1}{\beta}} \rangle$ containing the term $(1 - s_l)^{\frac{1}{\beta}}$ is kept instead of a calculated number results.

	$\langle \alpha < s_l \rangle$	$\langle \alpha \leq s_l \rangle$
$\langle \alpha \geq s_l \rangle$	$\top$	$\bot$
$\langle \alpha > s_l \rangle$	$\top$	$\top$
$\langle \alpha \geq (1 - s_l) \rangle$	$\top$	$\top$
$\langle \alpha > (1 - s_l) \rangle$	$\top$	$\top$

	$\langle \alpha < (1 - s_l) \rangle$	$\langle \alpha \leq (1 - s_l) \rangle$
$\langle \alpha \geq s_l \rangle$	$\bot$	$\bot$
$\langle \alpha > s_l \rangle$	$\bot$	$\bot$
$\langle \alpha \geq (1 - s_l) \rangle$	$\top$	$\bot$
$\langle \alpha > (1 - s_l) \rangle$	$\top$	$\top$

	$\langle \alpha < s_u \rangle$	$\langle \alpha \leq s_u \rangle$
$\langle \alpha \geq s_u \rangle$	$\top$	$\bot$
$\langle \alpha > s_u \rangle$	$\top$	$\top$
$\langle \alpha \geq (1 - s_u) \rangle$	$\bot$	$\bot$
$\langle \alpha > (1 - s_u) \rangle$	$\bot$	$\bot$

	$\langle \alpha < (1 - s_u) \rangle$	$\langle \alpha \leq (1 - s_u) \rangle$
$\langle \alpha \geq s_u \rangle$	$\top$	$\top$
$\langle \alpha > s_u \rangle$	$\top$	$\top$
$\langle \alpha \geq (1 - s_u) \rangle$	$\top$	$\bot$
$\langle \alpha > (1 - s_u) \rangle$	$\top$	$\top$

Table 2: The conditions, under which a row-column pair of symbolic fuzzy constraints is conjugated in the symbolic completion, where $\top$ means always and $\bot$ never.

It is easy to see that calculus has the *termination property*, i.e., the rules can only be applied finitely often to a finite set of constraints: Because the fuzzy assertions added by the rules to the set have smaller term sizes than the antecedents, and each

[4] This is because $(\textit{mol mol}\, B \sqcup \neg \textit{very very}\, B)^{\mathcal{I}} \in [0.877, 1]$ for all $B^{\mathcal{I}}$.

rule can be applied only once (or finitely often in the case of $(\forall_{\geq})$ and $(\exists_{\leq})$) to the same antecedent, there can be no infinite chain of rule applications.

Now let us see what happens when we apply the symbolic completion procedure to a set of fuzzy assertions like $\mathcal{S} = \{\langle a : C \geq s\rangle, \langle a : D < s\rangle\}$ which is obtained from a subsumption problem:

Lemma 1. Let $\mathcal{S}$ be a set of symbolic fuzzy assertions, where either all right hand sides of fuzzy assertions are s_l, or all right hand sides of fuzzy assertions are s_u. A symbolic completion of $\mathcal{S}$ contains only fuzzy assertions with the right hand sides s_l, $(1-s_l)$, s_l^θ, or $(1-s_l)^\theta$, respectively s_u, $(1-s_u)$, s_u^θ, or $(1-s_u)^\theta$.

Proof. We consider only the case that all right hand sides of fuzzy assertions in $\mathcal{S}$ are s_l. The other case is similar.

Because concept modifiers are applied only to primitive concepts, one can prove by a straightforward induction on the application of the rules that for each assertion $\langle \alpha \circ n\rangle$ (where $\circ$ is one of $>, \geq, \leq, <$) that is generated by a rule we have that n is one of s_l or $1-s_l$ when α is not a term like $x : A$ with a primitive concept A, and n is one of s_l, $1-s_l$, s_l^θ or $(1-s_l)^\theta$ for assertions referring to primitive concepts. (Observe that on the right hand sides of rules there are never primitive concepts.) □

Consider two symbolic assumptions $\langle a : C \geq s_l^{\theta_1}\rangle$ and $\langle a : C < (1-s_l)^{\theta_2}\rangle$. For what values of s_l do these clash? To determine this, we have to solve the in-equation $m^{\theta_1} > (1-m)^{\theta_2}$ for m within the interval $[0, 1]$. It is easy to see, that this holds whenever $m \in (\gamma, 1]$, where $\gamma \in [0, 1]$ is the value for that $\gamma^{\theta_1} = (1-\gamma)^{\theta_2}$. Then, the values of s_l doing the clash are in the interval $(\gamma, 1] \cap (0, 0.5]$. However, usually it is not easy to calculate this value γ. But since we are not interested in the precise values of n, but are only interested whether a clash occurs for all $m \in (0, 1]$ or not, we map the interval $(0, 1)$ to $(0, \infty)$ with the strictly monotonic function $f(m) = \frac{\log(1-m)}{\log(m)}$, and check whether a clash occurs for all values of $f(m) \in (0, \infty)$. (The values 0 and 1 will be considered separately.) The discussed in-equation $m^{\theta_1} > (1-m)^{\theta_2}$, for instance, is fulfilled whenever $f(m) \in (\frac{\theta_1}{\theta_2}, \infty)$.

Now let us formally describe the conditions, under which a set of has no clash-free completion.

Definition 3. Let $\mathcal{S}$ be a set of symbolic assertions in which either all right hand sides are s_l, or all right hand sides are s_u. A symbolic completion of a $\mathcal{S}$ is *closed* if it contains a pair of fuzzy assertions that clashes independently of $f(m)$ according to Table 3, or if the disjunction of the clash intervals for $f(m)$ according to Table 3 contains $(0, 1)$ when $\mathcal{S}$ refers to s_l, or $(1, \infty)$ when $\mathcal{S}$ refers to s_u.[5]

	$\langle \alpha < s^{\theta_2}\rangle$	$\langle \alpha \leq s^{\theta_2}\rangle$
$\langle \alpha \geq s^{\theta_1}\rangle$	$\theta_2 \geq \theta_1$	$\theta_2 > \theta_1$
$\langle \alpha > s^{\theta_1}\rangle$	$\theta_2 \geq \theta_1$	$\theta_2 \geq \theta_1$
$\langle \alpha \geq (1-s)^{\theta_1}\rangle$	$f(m) \in (0, \frac{\theta_2}{\theta_1}]$	$f(m) \in (0, \frac{\theta_2}{\theta_1})$
$\langle \alpha > (1-s)^{\theta_1}\rangle$	$f(m) \in (0, \frac{\theta_2}{\theta_1}]$	$f(m) \in (0, \frac{\theta_2}{\theta_1}]$
	$\langle \alpha < (1-s)^{\theta_2}\rangle$	$\langle \alpha \leq (1-s)^{\theta_2}\rangle$
$\langle \alpha \geq s^{\theta_1}\rangle$	$f(m) \in [\frac{\theta_1}{\theta_2}, \infty)$	$f(m) \in (\frac{\theta_1}{\theta_2}, \infty)$
$\langle \alpha > s^{\theta_1}\rangle$	$f(m) \in [\frac{\theta_1}{\theta_2}, \infty)$	$f(m) \in [\frac{\theta_1}{\theta_2}, \infty)$
$\langle \alpha \geq (1-s)^{\theta_1}\rangle$	$\theta_2 \geq \theta_1$	$\theta_2 > \theta_1$
$\langle \alpha > (1-s)^{\theta_1}\rangle$	$\theta_2 \geq \theta_1$	$\theta_2 \geq \theta_1$

Table 3: The conditions under which a row-column pair of fuzzy constraints clashes. s is one of s_l, s_u, $f(m) = \frac{\log(1-m)}{\log m}$ and $m \in (0, 1)$ is the value s can be replaced with such that the constraints clash. The fuzzy assertions $\langle \alpha > s\rangle$, $\langle \alpha > (1-s)\rangle$ etc. are treated as $\langle \alpha > s^1\rangle$, $\langle \alpha > (1-s)^1\rangle$ etc. The conjugacy conditions of Table 2 are subsumed by these since m is restricted to $(0, 1)$ for s_l and $(1, \infty)$ for s_u.

Example 2. Consider the formulas $C =$ *very* $A \sqcap \neg A \sqcap$ *less* B and $D = B \sqcup \neg B$. Thus, we have to execute the symbolic completion procedure with
$\mathcal{S} = \{\langle a : C \geq s_l\rangle, \ \langle a : D < s_l\rangle\}$,
$\mathcal{S}' = \{\langle a : C \geq s_u\rangle, \ \langle a : D < s_u\rangle\}$.

The completion procedure applied to $\mathcal{S}$ will add the fuzzy assertions

$\langle a :$ *very* $A \sqcap \neg A \geq s_l\rangle$, $\langle a :$ *very* $A \geq s_l\rangle$,
$\langle a : A \geq s_l^{1/2}\rangle$, $\langle a : \neg A \geq s_l\rangle$, $\langle a : A \leq 1 - s_l\rangle$,
$\langle a :$ *less* $B \geq s_l\rangle$, $\langle a : B \geq s_l^2\rangle$, $\langle a : B < s_l\rangle$,
$\langle a : \neg B < s_l\rangle$, $\langle a : B > 1 - s_l\rangle$

[5]"Contain" is meant in the sense of including the interval $(0, 1)$ resp. $(1, \infty)$ completely. Since the intervals in Table 3 have the form $(0, \gamma)$, $(0, \gamma]$, $[\gamma, \infty]$, or (γ, ∞), this check amounts to finding the largest of the intervals starting from 0 and the largest of the intervals ending at ∞, and check whether one of they contains the whole of $(0, 1)$, resp. $(1, \infty)$, or verify that there is no gap between them.

This is the only symbolic completion of $\mathcal{S}$. Using Table 3 we find in $\widetilde{\mathcal{S}}$ the following pairs of assertions that might clash, together with the conditions under which they clash:

$$\langle a : A \geq s_l^{\frac{1}{2}} \rangle \text{ and } \langle a : A \leq 1 - s_l \rangle \\ \text{clash iff } f(m) \in [\tfrac{1}{2}, \infty) \ , \\ \langle a : B \geq s_l^2 \rangle \text{ and } \langle a : B < s_l \rangle \quad \text{clash iff } 1 > 2, \\ \langle a : B > 1 - s_l \rangle \text{ and } \langle a : B < s_l \rangle \\ \text{clash iff } f(m) \in (0, \tfrac{1}{1}] \ .$$

The condition $1 > 2$ is false, and thus there is no clash condition independent of $f(m)$. Hence, we have to form the disjunction of the intervals $[\frac{1}{2}, \infty) \cup (0, \frac{1}{1}] = (0, \infty)$. This contains the interval $(0, 1)$ for s_l. Thus, $\widetilde{\mathcal{S}}$ is closed. Similarly, one can find that the only symbolic completion $\widetilde{\mathcal{S}'}$ of $\mathcal{S}'$ is closed. As we will see later, this is sufficient to prove that C subsumes D.

As we will see in the next lemma, there is a strong connection between the the application of rules and the symbolic application of rules.

Lemma 2. Let $\mathcal{S}$ be a set of symbolic assertions in which either all right hand sides are s_l, resp. all right hand sides are s_u, let m be a number within $(0, 0.5)$ resp. $(0.5, 1)$ and let $\widehat{\mathcal{S}}$ be a set of symbolic fuzzy assertions that resulted from $\mathcal{S}$ by a number of symbolic rule applications. Then, there is a symbolic application of a rule in $\widehat{\mathcal{S}}$ that results in a set $\widehat{\mathcal{S}'}$ iff there is an application of a rule in $\widehat{\mathcal{S}}\{s_l/m\}$ resp. $\widehat{\mathcal{S}}\{s_u/m\}$ that results in $\widehat{\mathcal{S}'}\{s_l/m\}$ resp. $\widehat{\mathcal{S}'}\{s_u/m\}$.

Proof. To prove this we need to show that each rule is symbolically applicable in $\widehat{\mathcal{S}}$ iff it is applicable in $\widehat{\mathcal{S}}\{s_l/m\}$ resp. $\widehat{\mathcal{S}}\{s_u/m\}$, and that the result of the first application can be mapped to the result of the latter application by replacing s_l resp. s_u by m. We limit our presentation to the $(\forall_\geq)$ rule under the condition that $\mathcal{S}$ refers to s_l and thus $m \in (0, 0.5)$. The other proofs are similar.

Assume that $(\forall_\geq)$ is symbolically applicable in $\widehat{\mathcal{S}}$. Then, there are symbolic fuzzy assertions $\langle w_1 : \forall R.C \geq n \rangle$ and ψ in $\widehat{\mathcal{S}}$ such that ψ is conjugated to $\langle (w_1, w_2) : R \leq 1 - n \rangle$. Because of Lemma 1, ψ has a right hand side of either s_l or $(1 - s_l)$, and we have either $n = s_l$ or $n = (1 - s_l)$. Considering $m \in (0, 0.5)$ and thus[6] $m < 1 - m$, a case analysis for these four cases and the possibilities for the comparison operator in ψ shows that $\psi\{s_l/m\}$ is conjugated to $\langle (w_1, w_2) : R \leq 1 - n \rangle\{s_l/m\}$, such that $(\forall_\geq)$ is applicable in $\widehat{\mathcal{S}}\{s_l/m\}$ as well, and that the result $\widehat{\mathcal{S}} \cup \{\langle w_2 : C \geq n \rangle\}$ of the application in $\widehat{\mathcal{S}}$ is mapped by replacement of s_l through m to the result $\widehat{\mathcal{S}}\{s_l/m\} \cup \{\langle w_2 : C \geq n \rangle\}\{s_l/m\}$ of the application in $\widehat{\mathcal{S}}\{s_l/m\}$. By a similar case analysis, the other direction follows: if a rule is applicable in $\widehat{\mathcal{S}}\{s_l/m\}$, it is symbolically applicable in $\widehat{\mathcal{S}}$ as well, and the result of this application can be mapped to the result of the application in $\widehat{\mathcal{S}}\{s_l/m\}$. □

We can now describe the connection between the notion "closed" and satisfiability.

Proposition 2.
1. Let $\mathcal{S}$ be a set of symbolic assertions in which all right hand sides are s_l. All symbolic completions of $\mathcal{S}$ are closed iff we have that $\mathcal{S}\{s_l/m\}$ is unsatisfiable for all $m \in (0, 0.5)$, where $\mathcal{S}\{s/m\}$ is the set of fuzzy assertions resulting from $\mathcal{S}$ by replacing all occurrences of s by m.

2. Let $\mathcal{S}$ be a set of symbolic assertions in which all all right hand sides are s_u. All symbolic completions of $\mathcal{S}$ are closed iff we have that $\mathcal{S}\{s_u/m\}$ is unsatisfiable for all $m \in (0.5, 1)$.

We omit the proof of this proposition for space reasons, and give the main theorem to decide subsumption for our fragment of $\mathcal{ALC}_{FH}$:

Theorem 1. Let C and D be two terms of $\mathcal{ALC}_{FH}$, where concept modifiers are applied only to primitive concepts. C subsumes D iff every symbolic completion of $\{\langle a : C \geq s_l \rangle, \langle a : D < s_l \rangle\}$ and $\{\langle a : C \geq s_u \rangle, \langle a : D < s_u \rangle\}$ is closed.

Proof. Because of Proposition 2 we have that $\{\langle a : C \geq n \rangle, \langle a : D < n \rangle\}$ is unsatisfiable for $n \in (0, 0.5) \cup (0.5, 1)$. This holds for $n = 0.5$ and $n = 1$ as well, because interpretation functions of $\mathcal{ALC}_{FH}$ use only continuous functions (compare Table 1). Thus, $\{\langle a : C \geq n \rangle, \langle a : D < n \rangle\}$ is unsatisfiable for

[6] This is the reason of the use of the two symbols s_l and s_u: while for a number m either $m < 1 - m$ holds or not, we need to restrict the intervals for a symbol to decide this.

$n \in (0, 1]$. Because of Proposition 1 we can conclude Theorem 1. □

6 Conclusion

In this paper we have presented the fuzzy description logic with hedges $\mathcal{ALC}_{FH}$. It is a conservative extension of Fuzzy-$\mathcal{ALC}$ defined by Umberto Straccia in [9]. In [9] the classical description logic $\mathcal{ALC}$ is extended by interpreting concepts and relations as fuzzy sets, the correspondence of the fuzzy semantics and the semantics of the equivalent "crisp" $\mathcal{ALC}$ is discussed and decision procedures for the entailment and subsumption problem have been defined. We extend the fuzzy description logic given there by concept manipulators whose semantics is based on hedge algebras. We have also presented an algorithm for determining unsatisfiability for full $\mathcal{ALC}_{FH}$, and an algorithm for determining subsumption for the fragment of $\mathcal{ALC}_{FH}$.

[7] follows an alternative way to extend description logic by introducing a probabilistic uncertainty about the classical interpretation of all primitive concepts. However, unlike our and related approaches, this does not overcome the sometimes inappropriate hard yes/no distinctions of the logic.

We hope that this paper is the starting point of a fruitful combination of description logic, fuzzy logic and hedge algebras based on the idea of manipulating concepts by hedges, and in future we shall find good real-world applications in this field.

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