THE FUZZY DESCRIPTION LOGIC $\mathcal{ALC}_{\mathcal{FLH}}$

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ABSTRACT

In this paper, we present the fuzzy description logic $\mathcal{ALC}_{FLH}$. $\mathcal{ALC}_{FLH}$ is based on $\mathcal{ALC}_{FH}$, but linear hedges are used instead of exponential ones. This allows to solve the entailment and the subsumption problem in a fuzzy description logic, where arbitrary concepts and roles may be modified.

KEY WORDS

Description logic, fuzzy logic and systems, hedge algebras.

1 Motivation

In most applications of description logics that we are aware of, except [7] and [8], concepts are crisp unary relations, i.e., an object may or may not be an element of a particular concept. Likewise, in most description logics roles are crisp binary relations. On the other hand, in many real-world applications like, for example, intelligent e-commerce information is often vague and imprecise. Fuzzy set theory introduced by Zadeh (see e.g. [9]) provides an ability to denote non-crisp concepts, i.e., an object may belong to a certain degree (typically a real number from the interval $[0, 1]$) to a particular relation.

Humans typically use linguistic adverbs like *very*, *more or less*, etc. to distinguish, for example, between a customer who is interested in technical details and one who is very interested in these details. In [10] Zadeh introduces so-called linguistic hedges modifying the shape of a fuzzy set by transforming it into another. Hedge algebras were considered in [5, 4] to give an algebraic charactrization of linguistic hedges. They have been applied to fuzzy logic in various ways (see e.g. [3]).

In $\mathcal{ALC}_{FH}$ [1, 2] hedges were used to modify concepts in the fuzzy description logic $\mathcal{ALC}_F$[7], which itself is an extension of $\mathcal{ALC}$. There was, however, a main restriction, viz. that modifiers had to be restricted to primitive concepts in order to solve the subsumption problem. In this paper we overcome this restriction by introducing so-called *linear* hedges in Section 3. $\mathcal{ALC}_{FLH}$ is then defined as an extension of $\mathcal{ALC}_{FH}$ allowing to modify arbitray concepts and roles in Section 4. We specify two normal forms and show that each concept can be normalized in Section 5. The entailment and the subsumption problems are solved in Sections 6 and 7, respectively. A brief disussion concludes the paper in Section 8.

2 Preliminaries

The approach presented in this paper is based on $\mathcal{ALC}_{FH}$ [1, 2], which is an extension of $\mathcal{ALC}$ and $\mathcal{ALC}_F$ [7]. *Concepts* C and D are constructed by the rule

$$\begin{array}{rlll} C, D \rightarrow & A & \mid & \text{(Primitive concept)} \\ & \top & \mid & \text{(Top concept)} \\ & \bot & \mid & \text{(Bottom concept)} \\ & \neg C & \mid & \text{(Negation concept)} \\ & C \sqcap D & \mid & \text{(Conjunction concept)} \\ & C \sqcup D & \mid & \text{(Disjunction concept)} \\ & MC & \mid & \text{(Modifier concept)} \\ & \forall R.C & \mid & \text{(Universal quantity)} \\ & \exists R.C & & \text{(Existance quantity)} \end{array}$$

where A denotes primite concepts, R roles, and M modifiers. We interprete formulas as usual in a fuzzy setting by mapping concepts and roles onto membership functions. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. Then,

$$\begin{array}{rcl} A^{\mathcal{I}} & : & \Delta^{\mathcal{I}} \rightarrow [0,1] \\ R^{\mathcal{I}} & : & \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \\ \top^{\mathcal{I}}(d) & = & 1 \text{ for all } d \in \Delta^{\mathcal{I}} \\ \bot^{\mathcal{I}}(d) & = & 0 \text{ for all } d \in \Delta^{\mathcal{I}} \\ (\neg C)^{\mathcal{I}}(d) & = & 1 - C^{\mathcal{I}}(d) \\ (C \sqcap D)^{\mathcal{I}}(d) & = & \min\{C^{\mathcal{I}}(d), C^{\mathcal{I}}(d)\} \\ (C \sqcup D)^{\mathcal{I}}(d) & = & \max\{C^{\mathcal{I}}(d), C^{\mathcal{I}}(d)\} \\ (MC)^{\mathcal{I}}(d) & = & \eta_M(C^{\mathcal{I}}(d)) \\ (\forall R.C)^{\mathcal{I}}(d) & = & \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d)\}\} \\ (\exists R.C)^{\mathcal{I}}(d) & = & \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d,d'), C^{\mathcal{I}}(d)\}\} \end{array}$$

where η_M is used to modify a membership function and $d \in \Delta^{\mathcal{I}}$. Such modifiers, or *linguistic hedges*, were introduced by Zadeh in [10], where he also proposed to use exponent functions as hedges. In [1, 2] a function *exponent* has been specified, which applied to a modifier M computes an exponent β such that $\eta_M(x) = x^{\mathit{exponent}(M)} = x^{\beta}$, where $x \in [0, 1]$.

Fuzzy assertions are expressions of the form $\langle \alpha \circ n \rangle$, where $\circ \in \{>, \geq, \leq, <\}$, α is of type $a : C$ or $(a, b) : R$, and $n \in [0, 1]$. *Fuzzy terminological axioms* as well as the semantics for fuzzy assertions and terminological axioms are defined as usual.

3 Linear Hedges

The main idea proposed in this paper is the use of linear instead of exponential hedges. Linear hedges were first introduced in [6], where it has been shown that they have better algebraic and computational properties than existing approaches for a parametric representation of linguistic truth-values. Here we consider the following linear hedges: Given a modifier M and let $\beta = \mathit{exponent}(M)$ and $x \in [0,1]$, then

$$\eta_M(x) = \begin{cases} \frac{1}{\beta}x & \text{if } x \leq \frac{\beta}{\beta+1}, \\ 1 + \beta(x-1) & \text{otherwise.} \end{cases}$$

One should observe that its inverse function is

$$\eta_M^{-1}(x) = \begin{cases} \beta x & \text{if } x \leq \frac{1}{\beta+1}, \\ 1 + \frac{1}{\beta}(x-1) & \text{otherwise.} \end{cases}$$

Hence, η_M^{-1} is obtained from η_M by replacing β by $\frac{1}{\beta}$.

4 $\mathcal{ALC}_{FLH}$

We can now specify $\mathcal{ALC}_{FLH}$ as follows: *Concepts* are defined as in Section 2. *Roles* R are defined as

$$R \rightarrow Q \mid MR,$$

where Q are primitive roles and M are modifiers. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. Then,

$$\begin{array}{rcl} Q^{\mathcal{I}} & : & \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \\ (MR)^{\mathcal{I}}(d,d') & = & \eta_M(R^{\mathcal{I}}(d,d')) \end{array}$$

where η_M is a linear linguistic hedge as defined in Section 3.

For each modifier M we assume that there is an *inverse modifier* M^{-1} with $\eta_{M^{-1}} = \eta_M^{-1}$. In the view of semantics, this assumption will be useful for converting a concept in $\mathcal{ALC}_{FLH}$ into an equivalent normal form. It is easy to see that concepts can be expressed without using inverse modfiers. For instance, the concept $M^{-1}C$ is semanctically equal to $\neg M(\neg C)$. In the real world, one can understand the meaning of an inverse modifier in a similar way. For example, the linguistic hedges *very* and *not-very* can be viewed as inverse to each other. The meaning of *very old* and *not-very old* can be intuitively divided in the middle by the concept *old*, which corresponds to a semantical ordering of the concepts $M^{-1}C$, C, and MC.

5 Normalizing Concepts in $\mathcal{ALC}_{FLH}$

In this section we define two normal forms for concepts and show that each concept can be transformed into these forms.

Proposition 1 *The following semantic equivalences hold in $\mathcal{ALC}_{FLH}$:*

$$\begin{array}{rcl} M(\neg C) & \equiv & \neg M^{-1}C \\ M(C \sqcap D) & \equiv & M(C) \sqcap M(D) \\ M(C \sqcup D) & \equiv & M(C) \sqcup M(D) \\ M(\forall R.C) & \equiv & \forall M^{-1}R.MC \\ M(\exists R.C) & \equiv & \exists MR.MC \end{array}$$

A proof can be easily obtained taking into account that we are using linear hedges.

The *set of simple concepts* is the smallest set satisfying the following conditions: (i) each primitive concept is a simple concept; (ii) if X is a simple concept and M is a modifier, then MX is a simple concept. In other words, simple concepts are obtained from primitive concepts by prefixing them with a string of modifiers.

A concept C is said to be in *modifier normal form* (*MNF*) iff all modifiers occurring in C act only on simple concepts and roles. The following result is an immediate consequence of Proposition 1. It is obtained by pushing modifiers into the structures as much as possible.

Proposition 2 *For each concept in $\mathcal{ALC}_{FLH}$ there is a semantically equivalent one in MNF.*

A concept C is said to be in *negative modifier normal form* (*NMNF*) iff it is in *MNF* and all negation signs occurring in C act only on simple concepts or roles. One should observe that the concept $M_1 \neg M_2 A$ is not in *NMNF* because it is not in *MNF*; however, it is equivalent to $\neg M_1^{-1} M_2 A$ which is in *NMNF*.

Proposition 3 *For each concept in $\mathcal{ALC}_{FLH}$ there is a semantically equivalent one in NMNF.*

This result is shown by first transforming a given concept C into MNF C' and, thereafter, transforming C' into NMNF in the usual way while treating simple concepts as atoms. The complexity of the whole normalization process is polynomial wrt the length of input concept.

6 The Entailment Problem in $\mathcal{ALC}_{FLH}$

Let Σ be a set of fuzzy assertions. As in $\mathcal{ALC}_{FH}$, an entailment problem $\Sigma \models \langle \alpha \circ n \rangle$ is converted into the problem of whether $\Sigma \cup \{\langle \alpha \bullet n \rangle\}$ is unsatisfiable, where $\circ \in \{<, \leq, \geq, >\}$ and $\bullet$ is the inverse of $\circ$. The latter problem is solved by a tableau algorithm. As usual, we use propagation rules on the set of assertions (or constraints) to convert constraints into simpler ones. This process will terminate and result in a *completion set* to which no propagation rule can be applied. As we will show in Proposition 4 the set is unsatisfiable iff all resulting sets of constraints contain a clash, where clashes are defined as in [7]. In particular, the following table lists all possible pairs of clashing contraints:

	$\langle \alpha < n \rangle$	$\langle \alpha \leq n \rangle$
$\langle \alpha \geq m \rangle$	$n \geq m$	$n > m$
$\langle \alpha > m \rangle$	$n \geq m$	$n \geq m$

We turn to the specification of the propagation rules. Due to lack of space, we cannot present all of them. Considering the propagation rules of $\mathcal{ALC}_{FH}$, we present here only the changes and additions. Because we have changed the semantics of modifiers in $\mathcal{ALC}_{FLH}$, we also have to change the corresponding propagation rules. As before, let $\circ \in \{<, \leq, \geq, >\}$ and $\bullet$ be the inverse of $\circ$.

$$\langle w : MC \circ n \rangle \rightarrow \langle w : C \circ \eta_{M^{-1}}(n) \rangle \qquad (M_\circ)$$

Because $\mathcal{ALC}_{FLH}$ allows modified roles we need an additional propagation rule:

$$\begin{aligned} &\langle (w, w') : MR \circ n \rangle \\ &\qquad \rightarrow \langle (w, w') : R \circ \eta_{M^{-1}}(n) \rangle \qquad (M_\circ^R) \end{aligned}$$

The existence of complex roles in $\mathcal{ALC}_{FLH}$ forces us also to change the propagation rules $(\forall_\geq)$, $(\forall_>)$, $(\exists_\leq)$ and $(\exists_<)$ used in $\mathcal{ALC}_{FH}$ slightly. Let Q be a primitive role.

$$\langle w_1 : \forall R.C \geq n \rangle, \psi^c \rightarrow \langle w_2 : C \geq n \rangle, \qquad (\forall_\geq)$$

where $\psi = \langle (w_1, w_2) : Q < \eta_{M_k}(\ldots \eta_{M_1}(1 - n) \ldots) \rangle$, $R = M_1 \ldots M_k Q$, and ψ^c is the conjugated constraint of ψ.

$$\langle w_1 : \forall R.C > n \rangle, \psi^c \rightarrow \langle w_2 : C > n \rangle, \qquad (\forall_>)$$

where $\psi = \langle (w_1, w_2) : Q \leq \eta_{M_k}(\ldots \eta_{M_1}(1 - n) \ldots) \rangle$, $R = M_1 \ldots M_k Q$, and ψ^c is the conjugated constraint of ψ.

$$\langle w_1 : \exists R.C \leq n \rangle, \psi^c \rightarrow \langle w_2 : C \leq n \rangle, \qquad (\exists_\leq)$$

where $\psi = \langle (w_1, w_2) : Q \leq \eta_{M_k}(\ldots \eta_{M_1}(n) \ldots) \rangle$, $R = M_1 \ldots M_k Q$, and ψ^c is the conjugated constraint of ψ.

$$\langle w_1 : \exists R.C < n \rangle, \psi^c \rightarrow \langle w_2 : C < n \rangle, \qquad (\exists_<)$$

where $\psi = \langle (w_1, w_2) : Q < \eta_{M_k}(\ldots \eta_{M_1}(n) \ldots) \rangle$, $R = M_1 \ldots M_k Q$, and ψ^c is the conjugated constraint of ψ.

Proposition 4 *A finite set of constraints in $\mathcal{ALC}_{FLH}$ is unsatisfiable iff all its completion sets contain a clash.*

The proof of this proposition is analogous to the case of $\mathcal{ALC}_{FH}$ proved in [1]. Because we use the same way as [7, 1] to deal with the entailment problem, our given decision procedure is also PSPACE-hard. As in [1] the propagation rules for $\forall$ and $\exists$ would lead to an exponential explosion. In [7, 1] this problem was solved by using so-called *trace rules*. We have not yet investigated the use of trace rules in $\mathcal{ALC}_{FLH}$, but, due to the similarity of the propagation rules we believe it also works for our case.

To the end of this section, we like to demonstrate the proposed solution of the entailment problem in $\mathcal{ALC}_{FLH}$. We are modifying an example already discussed in [1]. Consider an e-commerce application where we like to determine whether a customer, who is interested in technical aspects, is likely to want a particular product. Formally, let Σ be defined as follows:

$$\langle p : (\neg \exists \mathit{very}\, \mathit{attr}.\mathit{very}\, \mathit{int}) \sqcup \mathit{wants} \geq 0.9 \rangle \qquad (1)$$

$$\langle \mathit{tech} : \mathit{int} \geq 0.8 \rangle \qquad (2)$$

$$\langle (p, \mathit{tech}) : \mathit{attr} \geq 0.7 \rangle \qquad (3)$$

Suppose we want to show that $\Sigma \models \langle p : \mathit{wants} \geq 0.9 \rangle$. In order to do so we negate the goal:

$$\langle p : \mathit{wants} < 0.9 \rangle, \qquad (4)$$

add it to Σ, and attempt to prove that $\Sigma \cup \{(4)\}$ is unsatisfiable.

It is easy to see that rule $(\sqcup_\geq)$ (taken from $\mathcal{ALC}_{FH}$) is appliable to (1). Because this rule is non-deterministic, there are two cases which need to be considered:
(i) Applying $(\sqcup_\geq)$ to (1), we obtain

$$\langle p : \mathit{wants} \geq 0.9 \rangle. \qquad (5)$$

This assertion clashes immediately with (4).
(ii) We also obtain

$$\langle p : \neg \exists \mathit{very}\, \mathit{attr}.\mathit{very}\, \mathit{int} \geq 0.9 \rangle. \qquad (6)$$

Applying rule $(\neg^R_\geq)$ to (6), we obtain

$$\langle p : \exists \mathit{very}\, \mathit{attr}.\mathit{very}\, \mathit{int} < 0.1 \rangle. \qquad (7)$$

In order to apply $(\exists_<)$ to (7) consider

$$\psi = \langle (p, \mathit{tech}) : \mathit{attr} < \eta_{very}(0.1) \rangle \qquad (8)$$

and let *exponent* (*very*) $= \beta = 2$. In this case $\eta_{very}(0.1) = \frac{0.1}{2} = 0.05$ and, thus, (8) can be simplified to

$$\psi = \langle (p, \mathit{tech}) : \mathit{attr} < 0.05 \rangle. \qquad (9)$$

ψ is conjugated to (3) and, consequently, $(\exists_<)$ can be applied to (7) yielding

$$\langle \mathit{tech} : \mathit{very}\, \mathit{int} < 0.1 \rangle \qquad (10)$$

Applying rule $M_<$ to (10) we obtain

$$\langle \mathit{tech} : \mathit{int} < \eta_{very}(0.1) \rangle. \qquad (11)$$

As before, $\eta_{very}(0.1) = 0.05$ and, hence, (11) can be simplified to

$$\langle \mathit{tech} : \mathit{int} < 0.05 \rangle \qquad (12)$$

Because (12) is conjugated to (2), we conclude thet $\Sigma \cup \{(4)\}$ is unsatisfiable. Consequently,

$$\Sigma \models \langle p : \mathit{wants} \geq 0.9 \rangle.$$

7 The Subsumption Problem in $\mathcal{ALC}_{FLH}$

We will approach the subsumption problem as in the case of $\mathcal{ALC}_{FH}$[2] or $\mathcal{ALC}_F$[7]. In a first step, all complex concepts are expanded, which leads to a subsumption problem over an empty set of terminological axioms, and, thereafter, the subsumption problem is reduced to an entailment problem.

Proposition 5 *Let C and D be two concepts in $\mathcal{ALC}_{FLH}$. Then $C \sqsubseteq_\emptyset D$ iff $\langle a : C \geq n\rangle \models \langle a : D \geq n\rangle$ for all $n \in (0, 1]$, where a is an arbitrary individual.*

The proof of this proposition is similar to the corresponding one in [7] and, thus, we omit it here. The entailent problems considered in Proposition 5 will be reduced to equivalent unsatisfiability problems of the form $S = \{\langle a : C \geq n\rangle, \langle a : D < n\rangle\}$ for all $n \in (0, 1]$, to which we can apply the tableau algorithm presented in Section 6. Wlog we assume that all concepts mentioned in this section are in MNF.

Before coming to the main result of this section we need another auxiliary proposition. In order to simplify our exposition in the following we use the following abbreviation: Let M be a modifier and *exponent* $(M) = \beta$. Then, by η_β and $\eta_{\frac{1}{\beta}}$ we denote η_M and $\eta_{M^{-1}}$, respectively.

Proposition 6 *Let C and D be two $\mathcal{ALC}_{FLH}$ concepts, $S = \{\langle a : C \geq n\rangle, \langle a : D < n\rangle\}$, and S' be obtained from S by applying some propagation rules. Then, every constraint in S' has one of the following forms:*

1. $\langle w \geq \eta_{\beta_l}(\ldots\eta_{\beta_1}(n)\ldots)\rangle$,
2. $\langle w \leq \eta_{\beta_l}(\ldots\eta_{\beta_1}(1-n)\ldots)\rangle$,
3. $\langle w < \eta_{\beta_l}(\ldots\eta_{\beta_1}(n)\ldots)\rangle$,
4. $\langle w > \eta_{\beta_l}(\ldots\eta_{\beta_1}(1-n)\ldots)\rangle$,

where $\beta_1, \ldots, \beta_l > 0$, $l \geq 0$ and w is the form of $a : C$ or $(x, y) : R$.

The proof can be done by induction on the length of the derivation generated by the propagation rules. Now we turn to our main result in this section, which specifies a method for solving the entailment problem in $\mathcal{ALC}_{FLH}$.

Proposition 7 *Let C and D be two concepts in $\mathcal{ALC}_{FLH}$ and $S = \{\langle a : C \geq n\rangle, \langle a : D < n\rangle\}$. The following holds:*

1. *There is a finite set of completion sets of S, to each of which an interval $(a, b] \subseteq (0, 1]$ is associated,*
2. *Let $\tilde{S}$ be a completion set of S, $(a, b]$ the interval associated with $\tilde{S}$, $x \in (a, b]$, and $\tilde{S}'$ be obtained from $\tilde{S}$ by replacing n by x. If $\tilde{S}'$ is unsatisfiable then $\tilde{S}$ is unsatisfiable for all $y \in (a, b]$.*
3. *$C \sqsubseteq_\emptyset D$ iff all completion sets of S are unsatisfiable.*

Proof Let $\langle S, (a, b]\rangle$ be a pair, where all constraints in S are in the form listed in Proposition 6 and $(a, b]$ is the interval associated with S which restricts the possible values for symbolic "n" in S. In order to check whether a propagation rule is applicable to our pair, we consider three cases depending on the type of the propagation rule applied to S: (a) non-deterministic rules consisting of $(\sqcup_\geq)$, $(\sqcup_>)$, $(\sqcap_\leq)$ and $(\sqcap_<)$, (b) conditioned rules consisting of $(\forall_\geq)$, $(\forall_>)$, $(\exists_\leq)$ and $(\exists_<)$, and (c) deterministic rules consisting of all remaining rules.

(a): A non-deterministic rule $\Phi \rightarrow \Psi_1|\Psi_2$ is applicable to $\langle S, (a, b]\rangle$ iff $S \supseteq \Phi$. In this case we obtain two new pairs $\langle S_1, (a, b]\rangle$ and $\langle S_2, (a, b]\rangle$ where $S_1 = S \cup \Psi_1$ and $S_2 = S \cup \Psi_2$. E.g., consider $(\sqcap_<)$. It is applicable to $\langle S, (a, b]\rangle$ iff S contains a constraint of the form $\langle x : C \sqcap D < n\rangle$ and $(\sqcap_<)$ has not been applied to this constraint before. If applied we obtain two new pairs, viz. $S \cup \{\langle x : C < n\rangle\}$ and $S \cup \{\langle x : D < n\rangle\}$. Likewise, we obtain the same result for the other rules of this type.

(b): A conditioned rule $\Phi \rightarrow \Psi$ if Γ is applicable to $\langle S, (a, b]\rangle$ iff $S \supseteq \Phi$ and Γ is satisfied. In some cases Γ will force us to devide $\langle S, (a, b]\rangle$ into $m \geq 1$ pairs $\langle S, (a_i, a_{i+1}]\rangle$ with $i = 1, \ldots, m$ where $(a, b] = (a_1, a_2]\dot\cup(a_2, a_3]\dot\cup\ldots\dot\cup(a_m, a_{m+1}]$.

In order to complete this part of the proof, we need to apply some properties of the linear hedges defined in Section 3

1. $1 - \eta_\beta(n) = \eta_{\frac{1}{\beta}}(1 - n)$
2. $f(n) = \eta_{\beta_l}(\ldots\eta_{\beta_1}(n)\ldots)$ is an increasing, continuous function, which increases from 0 to 1 when n runs from 0 to 1 and can be expressed explicitly by at most $l + 1$ linear functions.

One should observe that these properties do not hold if we had used exponential hedges.

As an example, consider the rule $(\forall_\geq)$. It is applicable to $\langle S, (a, b]\rangle$ iff S contains a constraint of the form

$$\langle x : \forall R.C \geq n\rangle, \tag{13}$$

in addition, either a constraint of the form

$$\langle (x, y) : Q \geq \eta_{\alpha_k}(\ldots\eta_{\alpha_1}(n)\ldots)\rangle \tag{14}$$

or a constraint of the form

$$\langle (x, y) : Q > \eta_{\alpha_k}(\ldots\eta_{\alpha_1}(1-n)\ldots)\rangle \tag{15}$$

according to Proposition 6, where $R = M_l \ldots M_1 Q$, $(\forall_\geq)$ has not been applied to (13) before and, moreover we have to find a constraint which is conjugated to

$$\langle (a, b) : R \leq 1 - n\rangle. \tag{16}$$

Transforming (14) and (15) using $(M_\circ^R)$ we obtain

$$\langle (x, y) : R \geq \eta_{\beta_l}(\ldots\eta_{\beta_1}(\eta_{\alpha_k}(\ldots\eta_{\alpha_1}(n)\ldots))\ldots)\rangle$$

and

$$\langle (x,y) : R > \eta_{\beta_l}(\ldots \eta_{\beta_1}(\eta_{\alpha_k}(\ldots \eta_{\alpha_1}(1-n)\ldots))\ldots)\rangle,$$

respectively. These constraints are conjugated to (16) iff

$$1-n < \eta_{\beta_l}(\ldots \eta_{\beta_1}(\eta_{\alpha_k}(\ldots \eta_{\alpha_1}(n)\ldots))\ldots) \tag{17}$$

and

$$1-n \leq \eta_{\beta_l}(\ldots \eta_{\beta_1}(\eta_{\alpha_k}(\ldots \eta_{\alpha_1}(1-n)\ldots))\ldots). \tag{18}$$

Inequation (17) can be effectively solved as follows: As mentioned before, $f(x) = \eta_{\beta_n}(\ldots \eta_{\beta_1}(x)\ldots)$ is a continuously increasing function from 0 to 1 when x runs from 0 to 1 and $f(x)$ can be composed from at most $k+l+1$ linear functions. This shows that we can solve effectively the equation

$$1-n = \eta_{\beta_l}(\ldots \eta_{\beta_1}(\eta_{\alpha_k}(\ldots \eta_{\alpha_1}(n)\ldots))\ldots). \tag{19}$$

Let n_m be its solution.

Case 1a: If $n_m \leq a$, then $(\forall_{\geq})$ is applicable to $\langle S, (a,b]\rangle$. We obtain the new pair $S \cup \{\langle y : C \geq n\rangle\}$, which is in the required form.

Case 2a: if $n_m > b$ then $(\forall_{\geq})$ is not applicable to $\langle S, (a,b]\rangle$.

Case 3a: Finally, if $n_m \in (a,b]$, then we divide the interval $(a,b]$ into $(a,n_m]$ and $(n_m,b]$. We also devide the pair $\langle S, (a,b]\rangle$ into $\langle S, (a,n_m]\rangle$ and $\langle S, (n_m,b]\rangle$. $(\forall_{\geq})$ is applicable to $\langle S, (n_m,b]\rangle$ and we obtain $S \cup \{\langle y : C \geq n\rangle\}$. However, $(\forall_{\geq})$ is not applicable to $\langle S, (a,n_m]\rangle$.

Inequation (18) can be solved in a similar way. (18) can be transformed into

$$n \geq \eta_{\frac{1}{\beta_l}}(\ldots \eta_{\frac{1}{\beta_1}}(\eta_{\frac{1}{\alpha_k}}(\ldots \eta_{\frac{1}{\alpha_1}}(n)\ldots))\ldots). \tag{20}$$

As in the case of (17) we can specify $\eta_{\frac{1}{\beta_l}}(\ldots \eta_{\frac{1}{\beta_1}}(\eta_{\frac{1}{\alpha_k}}(\ldots \eta_{\frac{1}{\alpha_1}}(n)\ldots))\ldots)$ as a function which is composed by at most $k+l+1$ linear function $a_i n + b_i$ where $n \in (\gamma_i, \gamma_{i+1}]$ and $\gamma_1 = 0$, $\gamma_{k+l+1} = 1$, for $i = 1..(k+l+1)$. When solving an equation of the form

$$n \geq sn + t \tag{21}$$

in an interval $(\gamma_l, \gamma_r]$ there are three cases to consider:

Case 1b: If $s = 1 \wedge t \neq 0$ or $s \neq 1 \wedge \frac{t}{1-s} \notin (\gamma_l, \gamma_r]$, then $n = sn + t$ has no solution in $(\gamma]$. One can check whether (21) holds or not by trying any value in $(\gamma_l, \gamma_r]$.

Case 2b: When $s = 1 \wedge t = 0$, them (21) holds in $(\gamma_l, \gamma_r]$.

Case 3b: Otherwise, $n = sn + t$ has a solution $\gamma_m \in (\gamma_l, \gamma_r]$. This means that (21) is true in $(\gamma_l, \gamma_m]$ and false in $(\gamma_m, \gamma_r]$ or vice versa. One can test this by choosing any value from $(\gamma_l, \gamma_m]$ and $(\gamma_m, \gamma_r]$.

By case analysis, we have found a partition $(a,b] = (a_1,a_2] \cup (a_2,a_3] \cup \ldots \cup (a_m, a_{m+1}]$, where for each interval, we know whether $(\forall_{\geq})$ is applicable or not. Thus, we devide $\langle S, (a,b]\rangle$ into m pairs $\langle S, (a_i, a_{i+1}]\rangle$. For each new pair, if $(\forall_{\geq})$ is applicable, we obtain a new pair $S \cup \{\langle y : C \geq n\rangle\}$.

(c): A deterministic rule $\Phi \rightarrow \Psi$ is applicable to $\langle S, (a,b]\rangle$ iff $S \supseteq \Phi$. In this case we obtain the the new pair $\langle S', (a,b]\rangle$ where $S' = S \cup \Psi$. E.g., consider $(\sqcap_{>})$. It is applicable to $\langle S, (a,b]\rangle$ iff S contains a constraint of the form $\langle x : C \sqcap D > n\rangle$ and $(\sqcap_{>})$ had not been applied to this constraint before. If it is applicable, then we obtain the set $S \cup \{\langle x : C > n\rangle, \langle x : D > n\rangle\}$. Likewise, we obtain the same result for the other rules of this type.

The above case analyis shows the way to find an applicable rule for a pair $\langle S, (a,b]\rangle$. When there is no such rule, we need to find whether there is any clash in this pair. Let $\langle \tilde{S}, (a,b]\rangle$ be a completion pair, such that no propagation rule can be applied to $\tilde{S}$ in $(a,b]$. If $\tilde{S}$ contains an unsatisfiable constraint, it is trivially true that $\tilde{S}$ is unsatisfiable in $(a,b]$. If $\tilde{S}$ contains a conjugated pair of constraints, then there are at most four kinds of clashing pairs according to Proposition 6:

Case 1: The pair

$$\langle w \geq \eta_{\beta_k}(\ldots \eta_{\beta_1}(n)\ldots)\rangle$$

and

$$\langle w \leq \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(1-n)\ldots)\rangle$$

clashes iff

$$\eta_{\beta_k}(\ldots \eta_{\beta_1}(n)\ldots) > \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(1-n)\ldots)$$

or

$$\eta_{\frac{1}{\alpha_1}}(\ldots \eta_{\frac{l}{\alpha_l}}(\eta_{\beta_k}(\ldots \eta_{\beta_1}(n)\ldots)\ldots)) > 1-n.$$

It is easy to see that the equation

$$\eta_{\frac{1}{\alpha_1}}(\ldots \eta_{\frac{l}{\alpha_l}}(\eta_{\beta_k}(\ldots \eta_{\beta_1}(n)\ldots)\ldots)) = 1-n$$

has only one solution $n_m \in (0,1]$. If $n_m \leq a$, i.e., the inequation holds for $n \in (a,b]$, then $\tilde{S}$ is unsatifiable for all $n \in (a,b]$. If $n_m > b$, i.e., the inequation does not hold for $n \in (a,b]$ then $\tilde{S}$ is satifiable for all $n \in (a,b]$. If $n_m \in (a,b]$, i.e., the inequation does not hold for $n \in (a,n_m]$ but holds for $n \in (n_m,b]$, then $\langle S, (a,b]\rangle$ is divided into $\langle S, (a,n_m]\rangle$ which is satisfiable over $(a,n_m]$ and $\langle S, (n_m,b]\rangle$ which is unsatisfiable over $(n_m,b]$.

Case 2: The pair

$$\langle w \geq \eta_{\beta_k}(\ldots \eta_{\beta_1}(n)\ldots)\rangle$$

and

$$\langle w < \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(n) \ldots)\rangle$$

clashes iff

$$\eta_{\beta_k}(\ldots \eta_{\beta_1}(n) \ldots) \geq \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(n) \ldots)$$

or

$$\eta_{\frac{1}{\alpha_1}}(\ldots \eta_{\frac{1}{\alpha_l}}(\eta_{\beta_k}(\ldots \eta_{\beta_1}(n) \ldots) \ldots)) \geq n.$$

Considering the equation

$$\eta_{\frac{1}{\alpha_1}}(\ldots \eta_{\frac{1}{\alpha_l}}(\eta_{\beta_k}(\ldots \eta_{\beta_1}(n) \ldots) \ldots)) = n,$$

and using a similar argument as used for (18), we can effectively build up a partition of $(a, b] = (a_1, a_2] \cup (a_2, a_3] \cup \ldots \cup (a_m, a_{m+1}]$, where for each interval in this partition, our inequation holds for all n in this interval or not. By deviding $\langle \tilde{S}, (a, b]\rangle$ into m pairs $\langle \tilde{S}, (a_i, a_{i+1}]\rangle$ with $i = 1..m$, we conclude that if the inquation holds in $(a_i, a_{i+1}]$, then the corresponding $\tilde{S}$ is unsatisfiable in $(a_i, a_{i+1}]$ and vice versa.

Case 3 and 4: The pair $\langle w > \eta_{\beta_k}(\ldots \eta_{\beta_1}(1 - n) \ldots)\rangle$, $\langle w \leq \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(1 - n) \ldots)\rangle$ and the pair $\langle w > \eta_{\beta_k}(\ldots \eta_{\beta_1}(1 - n) \ldots)\rangle$, $\langle w < \eta_{\alpha_l}(\ldots \eta_{\alpha_1}(n) \ldots)\rangle$ can be treated similarly to case 1 and 2.

It follows that eventually we obtain a set of completion sets with corresponding intervals. Let us start with the initial pair $\langle S, (0, 1]\rangle$. We will check whether a propagation rule can be applied. If this is the case, then we apply the rule and obtain at least one and at most finitely many new pairs. For each new pair, we repeat this process until no rule can be applied any longer. This process will terminate because the size of S is finite. Finally, we get a finite set of completion sets, which is checked for a clash. This proves the two first parts of Proposition 7. The final part 3. follows immediately because if all completion sets are unsatisfiable then the initial set S is unsatisfiable for all $n \in (0, 1]$ and vice versa. That finishes our proof of Proposition 7. ■

Furthermore, because the way used to solve the problem is similar to the case of the entailment problem, the decision procedure for subsumption problem has the same complexity.

8 Conclusion

In this paper we have introduced the fuzzy description logic $\mathcal{ALC}_{FLH}$, where concepts and roles are modified using linear hedges, and have solved the entailment as well as the subsumption problem. This extends previous work [1, 2], where modifiers were restricted to primitive concepts in order to solve the subsumption problem. We are currently in the process of building in trace rules in order to avoid an exponential explosion when applying the propagation rules for the quantifiers. We are also working on a prototypical implementation, which is the basis for running real-world examples.

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