The Fuzzy Linguistic Description Logic $\mathcal{ALC}_{FL}$

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Abstract

We present the fuzzy linguistic description logic $\mathcal{ALC}_{FL}$, an instance of the description logic framework $\mathcal{L}$-$\mathcal{ALC}$ with the certainty lattice characterized by a hedge algebra. Beside constructors of $\mathcal{L}$-$\mathcal{ALC}$, $\mathcal{ALC}_{FL}$ allows the modification by hedges.

Keywords: Description logics, hedge algebras, uncertainty.

1 Introduction

Description Logics (DLs) have been studied and applied successfully in quite a lot of fields (see e.g. [1]). To deal with vague and imprecise information in real-world applications, fuzzy $\mathcal{ALC}$ [8] introduces fuzzy concepts. As a more general case of fuzzy $\mathcal{ALC}$, a DL framework $\mathcal{L}$-$\mathcal{ALC}$ based on certainty lattices is presented in [9].

Humans typically use linguistic modifiers (hedges) like "very", "more or less" etc. to distinguish, e.g. between an old man and a very old one. In [10] Zadeh uses exponent functions to represent hedges modifying fuzzy sets, e.g. $\mu_{veryA}(u) = \mu_A(u)^2$. In many human languages, there is almost a continuum of phrases like "more or less", "much less", "possibly rather" and so forth expressing different levels of emphasis. Hedge Algebras (HAs), introduced in [6], give an algebraic characterization of such linguistic hedges. $\mathcal{ALC}_{FH}$ [4] and $\mathcal{ALC}_{FLH}$ [3] extend fuzzy $\mathcal{ALC}$ by allowing the modification by hedges of HAs.

In general, the domain of a HA can be represented as a lattice. Thus, an instance of $\mathcal{L}$-$\mathcal{ALC}$, where the certainty lattice is a truth domain represented by a HA, is a DL in which the truth degree of an assertion is a linguistic value, e.g., $John$ is an element of a concept $Young$ with degree $VeryTrue$. The idea is meaningful because in daily life, when being asked to assess the degree of a person being $Young$, it is usually easier to give a verbal answer like, for example, $Very\_High$ or $Quite\_True$, rather than to give a numerical answer like, for example, 0.5 or 0.7.

In this paper, we present the fuzzy linguistic DL $\mathcal{ALC}_{FL}$, which is such an instance of $\mathcal{L}$-$\mathcal{ALC}$. Beside constructors of $\mathcal{L}$-$\mathcal{ALC}$, $\mathcal{ALC}_{FL}$ allows the modification by hedges. Because the certainty lattice is characterized by a HA, the modification by hedges becomes more natural than that in $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$. Moreover, we show that $\mathcal{ALC}_{FL}$ overcomes the following drawback in $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$.

In $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$, the hedge application is ambiguous. For example, the concept $VeryMolYoung$[1] can be interpreted as $(VeryMol)Young$ in which $VeryMol$ is a modifier, or $Very(MolYoung)$ in which $Very$ and Mol are two different modifiers. Unfortunately, in both $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$ $(VeryMol)Young \neq Very(MolYoung)$. Therefore, we may have

$$\langle a : Very(MolYoung) < 0.7\rangle \not\models \langle a : (VeryMol)Young < 0.7\rangle,$$

[1] Mol is an abbreviation for more-or-less

which is surprising.

The paper is structured as follows. In Section 2, we discuss linear symmetric HAs and restrict ourselves to monotonic HAs. Then, $\mathcal{ALC}_{FL}$ is introduced in Section 3 and the satisfiability problem is discussed in Section 4. A brief conclusion in Section 5 concludes the paper.

2 Logical Basis

2.1 Linear Symmetric Hedge Algebras

In order to define inverse mappings of hedges later on with ease, we consider linear symmetric HAs only. The readers are referred to [5, 6, 7] for general HAs.

Consider a truth domain consisting of linguistic values, e.g., $VeryVeryTrue$, $PossiblyMoreFalse$, etc. In such a truth domain the value $VeryVeryTrue$ is obtained by applying the modifier $Very$ twice to the generator $True$. Thus, given a set of generators $G = \{True, False\}$ and a nonempty finite set H of hedges, the set X of linguistic values is $\{\delta c \mid c \in G,\ \delta \in H^*\}$. Furthermore, if we consider $True > False$, then this order relation also holds for other pairs, e.g., $VeryTrue > MoreTrue$. It means that there exists a partial order $>$ on X.

In general, given nonempty finite sets G and H of generators and hedges resp., the set of values generated from G and H is defined as $X = \{\delta c \mid c \in G,\ \delta \in H^*\}$. Given a strictly partial order $>$ on X, we define $u \geq v$ iff $u > v$ or $u = v$. Thus, X is described by an abstract algebra $AX = (X, G, H, >)$.

Each hedge $h \in H$ can be regarded as a unary function $h : X \to X,\ x \mapsto hx$. Moreover, suppose that each hedge is an *ordering operation*, i.e., $\forall h \in H.\ \forall x \in X.\ hx > x$ xor $hx < x$. Let $I \notin H$ be the identity hedge, i.e., $Ix = x$ for all $x \in X$. A hedge chain σ is a word over H, $\sigma \in H^*$. In a hedge chain $h_p \dots h_1$, h_1 is called the first hedge whereas h_p is called the last one. For $h, k \in H$, h and k are *converse* if $\forall x \in X.\ hx > x$ iff $kx < x$; h and k are *compatible* if $\forall x \in X.\ hx > x$ iff $kx > x$; h modifies terms stronger or equal than k, denoted by $h \geq k$, if $\forall x \in X.\ (hx \leq kx \leq x)$ or $(hx \geq kx \geq x)$; $h > k$ if $h \geq k$ and $h \neq k$; h is *positive wrt* k if $\forall x \in X.\ (hkx < kx < x)$ or $(hkx > kx > x)$; h is *negative wrt* k if $\forall x \in X.\ (kx < hkx < x)$ or $(kx > hkx > x)$.

The most commonly used HAs are symmetric ones, in which there are exactly two generators, like e.g., $G = \{True, False\}$. In this paper, we only consider symmetric HAs. Let $G = \{c^+, c^-\}$ where $c^+ > c^-$. c^+ and c^- are called *positive* and *negative generators* respectively. The set H is decomposed into the subsets $H^+ = \{h \in H \mid hc^+ > c^+\}$ and $H^- = \{h \in H \mid hc^+ < c^+\}$. For each value $x \in X$, let $H(x) = \{\sigma x \mid \sigma \in H^*\}$.

An abstract algebra $AX = (X, G, H, >)$, where $H \neq \emptyset$, $G = \{c^+, c^-\}$ and $X = \{\sigma c \mid c \in G, \sigma \in H^*\}$, is called a *linear symmetric HA* if it satisfies the following conditions:

- (A1) For all $h \in H^+$ and $k \in H^-$, h and k are converse.
- (A2) The sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered with the least element I.
- (A3) For each pair $h, k \in H$, either h is positive or negative wrt k.
- (A4) If $h \neq k$ and $hx < kx$ then $h'hx < k'kx$, for all $h, k, h', k' \in H$ and $x \in X$.
- (A5) If $u \notin H(v)$ and $u < v$ ($u > v$) then $u < hv$ ($u > hv$, resp.), for any $h \in H$.

Example 1. *Consider a HA $AX = (X, \{True, False\}, H, >)$, where $H = \{Very, More, Probably, Mol\}$, and (i) $Very$ and $More$ are positive wrt $Very$ and $More$, negative wrt $Probably$ and Mol; (ii) $Probably$ and Mol are negative wrt $Very$ and $More$, positive wrt $Probably$ and Mol.*

H is decomposed into $H^+ = \{Very, More\}$ and $H^- = \{Probably, Mol\}$. In $H^+ \cup \{I\}$ we have $Very > More > I$, whereas in $H^- \cup \{I\}$ we have $Mol > Probably > I$.

Proposition 2 ([6]). *For a linear symmetric HA $AX = (X, G, H, >)$, X is linearly ordered.*

Given $x = \sigma c$, where $\sigma \in H^*, c \in \{c^+, c^-\}$, we call $y = \sigma c'$ the *contradictory element* of x, denoted by $y = -x$, if $\{c, c'\} = \{c^+, c^-\}$. Let

$x, y \in X$, we define $\vee$, $\wedge$, and $\rightarrow$ as: $x \vee y = \max(x, y)$; $x \wedge y = \min(x, y)$; $x \rightarrow y = -x \vee y$.

In [7], HAs are extended by adding two artificial hedges inf and sup defined as $\inf(x) = \text{infimum}(H(x))$, $\sup(x) = \text{supremum}(H(x))$. If $H \neq \emptyset$, $H(c^+)$ and $H(c^-)$ are infinite, according to [7] $\inf(c^+) = \sup(c^-)$. Let $W = \inf(c^+) = \sup(c^-)$, $\sup(True) = 1$, $\inf(False) = 0$, i.e., 1 and 0 resp. are the greatest and the least elements of X. The following properties show that X can be used as the truth domain for a non-classical logic.

Proposition 3 ([7]). *For every symmetric extended HA, the following properties hold:*

1. $-hx = h(-x)$, *for any* $h \in H$;
2. $- - x = x$; $-1 = 0$, $-0 = 1$, $-W = W$;
3. $-(x \vee y) = (-x \wedge -y)$,
   $-(x \wedge y) = (-x \vee -y)$;
4. $x \wedge -x < W < y \vee -y$;
5. $x > y$ *iff* $-x < -y$;
6. $x \rightarrow y = -y \rightarrow -x$;
7. $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
8. $x \rightarrow y \geq x' \rightarrow y'$
   if $x \leq x'$ *and/or* $y \geq y'$;
9. $1 \rightarrow x = x$, $x \rightarrow 1 = 1$,
   $0 \rightarrow x = 1$, $x \rightarrow 0 = -x$;
10. $x \rightarrow y > W$ *iff* $x < W$ *or* $y > W$;
11. $x \rightarrow y < W$ *iff* $y < W$ *and* $x > W$;
12. $x \rightarrow y = 1$ *iff* $x = 0$ *or* $y = 1$.

To define the semantics of the hedge modification in our logic, we define the so-called *inverse mapping* of a hedge. In order to define it with ease, let us consider some restrictions for the HAs representing the truth domain. In the rest of the paper, without stating otherwise, "hedge algebra" means "linear symmetric hedge algebra".

2.2 Hedge Algebras as Truth Domains

A HA $AX = (X, G, H, >)$ is called *monotonic* if each $h \in H^+$ (H^-) is positive wrt all $k \in H^+$ (H^-), and negative wrt all $k \in H^-$ (H^+).

As defined, both sets $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ are linearly ordered. However, $H \cup \{I\}$ is not, e.g., in Example 1 $Very \in H^+$ and $Mol \in H^-$ are not comparable. Let us extend the order relation on $H^+ \cup \{I\}$ and $H^- \cup \{I\}$ to one on $H \cup \{I\}$ as follows: Given $h, k \in H \cup \{I\}$, $h \geq_h k$ iff $h \in H^+, k \in H^-$; or $h, k \in H^+ \cup \{I\}$ and $h \geq k$; or $h, k \in H^- \cup \{I\}$ and $h \leq k$. $h >_h k$ iff $h \geq_h k$ and $h \neq k$.

Example 4. *The HA in Example 1 is monotonic. The order relation $>_h$ in $H \cup \{I\}$ is $Very >_h More >_h I >_h Probably >_h Mol$.*

Then, in monotonic HAs, hedges are "context-free", i.e., a hedge modifies the meaning of a linguistic value independently of preceding hedges in the hedge chain.

Proposition 5. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then,*

$$h >_h k \quad \Leftrightarrow \quad h\sigma c^+ > k\sigma c^+ \qquad (1)$$

Proof. By induction on the length of σ:

Base step: $\sigma = \epsilon$: obvious.

Induction step: Assume that (1) holds for $\sigma = h_n \ldots h_1$ and consider $\sigma' = h_{n+1}h_n \ldots h_1$.

(⇒) Assuming $h >_h k$, we consider 2 cases:

(i) $h_{n+1}h_n \ldots h_1c^+ > h_n \ldots h_1c^+$: because (1) holds for σ, $h_{n+1}h_n \ldots h_1c^+ > h_n \ldots h_1c^+ = Ih_n \ldots h_1c^+$ implies $h_{n+1} >_h I$ and thus $h_{n+1} \in H^+$. There are 3 cases for h and k:

- $h \in H^+, k \in H^-$: because AX is monotonic, h is positive w.r.t h_{n+1} whereas k is negative w.r.t h_{n+1}. Therefore, $hh_{n+1} \ldots h_1c^+ > h_{n+1} \ldots h_1c^+ > kh_{n+1} \ldots h_1c^+$, i.e., (⇒) holds for σ'.
- $h, k \in H^+ \cup \{I\}$ and $h > k$, i.e., h, k are positive w.r.t h_{n+1}. Because $h > k$, $hh_{n+1} \ldots h_1c^+ > kh_{n+1} \ldots h_1c^+ \geq h_{n+1} \ldots h_1c^+$, i.e., (⇒) holds for σ'.
- $h, k \in H^- \cup \{I\}$ and $h < k$: similar to the previous case.

(ii) $h_{n+1}h_n \ldots h_1c^+ < h_n \ldots h_1c^+$: similar to the previous case.

(⇐) Assume $h\sigma c^+ > k\sigma c^+$. Suppose $h \leq_h k$, then either $h <_h k$ or $h = k$. If $h <_h k$, the proof of (⇒) implies $h\sigma c^+ < k\sigma c^+$. If $h = k$

then $h\sigma c^+ = k\sigma c^+$. Thus, $h\sigma c^+ \leq k\sigma c^+$, which contradicts the assumption.

Hence, (1) holds for σ of length n implies (1) holds for σ' of length $n+1$. Consequently, (1) holds for all $\sigma \in H^*$. □

In Proposition 5, the last hedge is independent of the others. Conversely, when being the first hedge, it does not affect the meaning of the others.

Proposition 6. *Given a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then, $\forall h \in H$: $\sigma_1 c^+ > \sigma_2 c^+ \Leftrightarrow \sigma_1 h c^+ > \sigma_2 h c^+$.*

Proof. Let $\sigma_1 = h_p \dots h_1$, $\sigma_2 = k_q \dots k_1$. Because $\sigma_1 c^+ > \sigma_2 c^+$, we have $\sigma_1 \neq \sigma_2$. Thus, there exists $n \geq 0$ such that $h_n \dots h_1 = k_n \dots k_1$ and $h_{n+1} \neq k_{n+1}$. We have

$$\begin{aligned} & h_p \dots h_1 c^+ > k_q \dots k_1 c^+ \\ \Leftrightarrow\ & h_{n+1} \dots h_1 c^+ > k_{n+1} \dots k_1 c^+ \\ \Leftrightarrow\ & h_{n+1} >_h k_{n+1} \\ \Leftrightarrow\ & h_{n+1} h_n \dots h_1 h c^+ > k_{n+1} k_n \dots k_1 h c^+ \\ \Leftrightarrow\ & h_p \dots h_1 h c^+ > k_q \dots k_1 h c^+ \end{aligned}$$

Hence, $\sigma_1 c^+ > \sigma_2 c^+ \Leftrightarrow \sigma_1 h c^+ > \sigma_2 h c^+$. □

In the general case, when the generator is either c^+ or c^-, a similar property holds.

Proposition 7. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. We have $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 h c_1 > \sigma_2 h c_2$, for $c_1, c_2 \in \{c^+, c^-\}$.*

The following corollary follows immediately.

Corollary 8. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$. Then, $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 \delta c_1 > \sigma_2 \delta c_2$.*

Therefore, in monotonic HAs, a hedge is not only independent of other hedges in the hedge chain but also independent of the generators, if it is the first or the last hedge in the chain. This property will help us to define the concept of inverse hedges in the next subsection.

2.3 Inverse Mappings of Hedges

In daily life, people often use words in relative assessments, e.g., "it is quite true that Robert is very old". As discussed in [10], assessments like that can be considered as a composition of an individual, e.g., Robert, a fuzzy predicate, e.g., VeryOld, and a truth value, e.g., QuiteTrue. In the context of fuzzy DLs, the above assessment is typically represented by

$$(VeryOld)^{\mathcal{I}}(Robert^{\mathcal{I}}) = QuiteTrue.$$

In a fuzzy linguistic logic [10], the following two assessments are equivalent: "it is true that Robert is very old" and "it is very true that Robert is old". It means that somehow the modifier from the truth value can be moved to the fuzzy predicate and vice versa. This idea is formalized in [5] by the two following rules represented in a DL representation:

$$\begin{aligned} RT1: \quad & (hC)^{\mathcal{I}}(a) = \sigma c \rightarrow C^{\mathcal{I}}(a) = \sigma h c \\ RT2: \quad & C^{\mathcal{I}}(a) = \sigma h c \rightarrow (hC)^{\mathcal{I}}(a) = \sigma c \end{aligned}$$

in which C is a concept, hC a concept C modified by a hedge h, a an individual, $\mathcal{I}$ an interpretation, σ a hedge chain, and c a generator of the truth domain.

However, the rules are not complete. E.g., if the truth-degree of "John is Young" is VeryTrue and we want to compute the truth-degree of "John is MoreYoung", then no rules are applicable. This problem motivates the definition of a so-called *inverse mapping of a hedge*, and based on this definition, a generalized version of rule (RT2).

Suppose that for each hedge $h \in H$, there exists a mapping $h^- : X \rightarrow X$ such that $h^-(\sigma h c) = \sigma c$ for all $\sigma \in H^*, c \in G$. Then the rule $(RT2)$ is generalized as follows:

$$GRT2 : C^{\mathcal{I}}(a) = \delta c \rightarrow (hC)^{\mathcal{I}}(a) = h^-(\delta c).$$

Note that $GRT2$ becomes $RT2$ when $\sigma h = \delta$.

In the following, we define h^- formally by axiomization. Given a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$ and a hedge $h \in H$. Suppose $h^- : X \rightarrow X$ is a mapping such that

$$h^-(\sigma h c) = \sigma c, \qquad \text{for } c \in \{c^+, c^-\} \qquad (2)$$

According to Propostion 7, $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow \sigma_1 h c_1 > \sigma_2 h c_2$. By (2), $h^-(\sigma_1 h c_1) =$

$\sigma_1 c_1$, $h^-(\sigma_2 h c_2) = \sigma_2 c_2$. Hence, $h^-(\sigma_1 h c_1) > h^-(\sigma_2 h c_2) \Leftrightarrow \sigma_1 h c_1 > \sigma_2 h c_2$. Generalizing this idea, h^- should satisfy:

$$\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow h^-(\sigma_1 c_1) > h^-(\sigma_2 c_2) \quad (3)$$

Therefore, a mapping $h^- : X \to X$ is called an *inverse mapping* of h iff it satisfies (2) and (3). A question concerning the existence of such mappings can be raised. Let us consider the following example.

Example 9. *Consider the HA given in Example 1. For $H(True)$, the inverse mappings are defined as follows*

$$V^-(\sigma T) = \begin{cases} \delta T & \text{if } \sigma = \delta V, \\ \sigma MolMolT & \text{otherwise.} \end{cases}$$

$$M^-(\sigma T) = \begin{cases} \sigma VVT & \text{if } \sigma = \delta V, \\ \delta T & \text{if } \sigma = \delta M, \\ \sigma MolMolT & \text{otherwise.} \end{cases}$$

$$P^-(\sigma T) = \begin{cases} \sigma MolMolT & \text{if } \sigma = \delta Mol, \\ \delta T & \text{if } \sigma = \delta P, \\ \sigma VVT & \text{otherwise.} \end{cases}$$

$$Mol^-(\sigma T) = \begin{cases} \delta T & \text{if } \sigma = \delta Mol, \\ \sigma VVT & \text{otherwise.} \end{cases}$$

in which T, V, M, and P stand for True, Very, More, and Probably resp.

For $H(False)$, the mappings are defined as $h^-(\sigma False) = -h^-(\sigma True)$ for each $h \in H$.

It is easily verified that these mappings satisfy (2) *and* (3).

Given hedges $h_1, \ldots, h_p$, one may need to construct an inverse mapping $(h_p \ldots h_1)^- : X \to X$ of a hedge chain $h_p \ldots h_1$. On the one hand, we expect $(h_p \ldots h_1)^-(\sigma h_p \ldots h_1 c) = \sigma c$. On the other hand, we have

$$\begin{aligned} & {h_p}^-(\ldots({h_2}^-({h_1}^-(\sigma h_p \ldots h_2 h_1 c)))\ldots) \\ = \; & {h_p}^-(\ldots({h_2}^-(\sigma h_p \ldots h_2 c))\ldots) \\ & \ldots \\ = \; & {h_p}^-(\sigma h_p c) = \sigma c \end{aligned}$$

Therefore, $(h_p \ldots h_1)^-$ can be defined as

$$(h_p \ldots h_1)^-(\sigma c) = {h_p}^-(\ldots({h_1}^-(\sigma c))\ldots) \quad (4)$$

We have the following property which is the general case of Corollary 8.

Proposition 10. *Consider a monotonic HA $AX = (X, \{c^+, c^-\}, H, >)$, a hedge chain δ and its inverse mapping δ^-. Then, $\sigma_1 c_1 > \sigma_2 c_2$ iff $\delta^-(\sigma_1 c_1) > \delta^-(\sigma_2 c_2)$.*

Proof. Let $\delta = h_p \ldots h_1$. According to (4), $\delta^-(\sigma c) = {h_p}^-(\ldots({h_1}^-(\sigma c))\ldots)$. According to (3), we have $\sigma_1 c_1 > \sigma_2 c_2 \Leftrightarrow {h_1}^-(\sigma_1 c_1) > {h_1}^-(\sigma_2 c_2) \Leftrightarrow \ldots \Leftrightarrow {h_p}^-(\ldots({h_1}^-(\sigma_1 c_1))\ldots) > {h_p}^-(\ldots({h_1}^-(\sigma_2 c_2))\ldots)$. □

3 $\mathcal{ALC}_{FL}$

This section discusses the *fuzzy linguistic description logic* $\mathcal{ALC}_{FL}$, i.e., a DL in which the truth domain of interpretations is represented by a hedge algebra.

The syntax of $\mathcal{ALC}_{FL}$ is similar to that of $\mathcal{L}$-$\mathcal{ALC}$ except that $\mathcal{ALC}_{FL}$ allows concept modifiers. Hence, $\mathcal{ALC}_{FL}$-concepts are defined by

$$A \,|\, \top \,|\, \bot \,|\, \neg C \,|\, C \sqcap D \,|\, C \sqcup D \,|\, \delta C \,|\, \exists R.C \,|\, \forall R.C,$$

where A denotes primitive concepts, R roles, C and D concepts, and δ modifiers.

The semantics is based on the notion of interpretations. Given a monotonic HA $AX = (X, \{True, False\}, H, >)$, an interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ in which $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^\mathcal{I}$ is a mapping, which maps different individuals to different elements in $\Delta^\mathcal{I}$, concept C to a function $C^\mathcal{I} : \Delta^\mathcal{I} \to X$ and role R to a function $R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to X$. The extension of $\mathcal{I}$ for complex concepts is

$$\begin{aligned} \top^\mathcal{I}(d) &= \sup(True) \text{ for all } d \in \Delta^\mathcal{I}, \\ \bot^\mathcal{I}(d) &= \inf(False) \text{ for all } d \in \Delta^\mathcal{I}, \\ (\neg C)^\mathcal{I}(d) &= -C^\mathcal{I}(d), \\ (C \sqcap D)^\mathcal{I}(d) &= C^\mathcal{I}(d) \wedge D^\mathcal{I}(d), \\ (C \sqcup D)^\mathcal{I}(d) &= C^\mathcal{I}(d) \vee D^\mathcal{I}(d), \\ (\delta C)^\mathcal{I}(d) &= \delta^-\left(C^\mathcal{I}(d)\right), \\ (\forall R.C)^\mathcal{I}(d) &= \textstyle\bigwedge_{d' \in \Delta^\mathcal{I}} (-R^\mathcal{I}(d, d') \vee C^\mathcal{I}(d')), \\ (\exists R.C)^\mathcal{I}(d) &= \textstyle\bigvee_{d' \in \Delta^\mathcal{I}} (R^\mathcal{I}(d, d') \wedge C^\mathcal{I}(d')), \end{aligned}$$

where $\wedge, \vee$ are the meet and join operations resp., $-x$ is the contradictory element of x, and δ^- is the inverse of the hedge chain δ.

Fuzzy assertions are expressions of the forms $\langle \alpha \circ x \rangle$ where $\circ \in \{>, \geq, \leq, <\}$, α is of type

$a : C$ or $(a, b) : R$, and $x \in X$. *Fuzzy terminological axioms*, the semantics of fuzzy assertions and terminological axioms are defined similarly to those in [3, 4, 8, 9].

The following semantic equivalences are proved easily because the HA in consideration is linear:

Proposition 11.

$$\begin{array}{lll} \delta(\neg C) & \equiv & \neg\delta(C), \\ \delta(C \sqcap D) & \equiv & \delta(C) \sqcap \delta(D), \\ \delta(C \sqcup D) & \equiv & \delta(C) \sqcup \delta(D), \\ \delta_1(\delta_2 C) & \equiv & (\delta_1\delta_2)C. \end{array}$$

For instance, $Very(MolC) \equiv (VeryMol)C$. Therefore the drawback of $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$ specified in Section 1 is solved in $\mathcal{ALC}_{FL}$.

4 The Satisfiability Problem

Similarly to fuzzy $\mathcal{ALC}$, $\mathcal{ALC}_{FH}$, $\mathcal{ALC}_{FLH}$, and $\mathcal{L}$-$\mathcal{ALC}$, in $\mathcal{ALC}_{FL}$ the entailment problem can be converted to the satisfiability problem, which can be solved by a tableau algorithm. As usual, starting from a set S of fuzzy constraints, the propagation rules are applied step by step to add "simpler" constraints preserving the satisfiability. This process terminates and gives a *completion set* to which no rules are applicable. If there is no clash in the completion set, we can construct a model for S, otherwise, S is unsatisfiable.

A set S of fuzzy constraints contains a *clash* iff it contains either one of the unsatisfiable constraints $\langle a : \bot \geq x\rangle$ where $x > 0$, $\langle a : \top \leq x\rangle$ where $x < 1$, $\langle \psi < 0\rangle, \langle \psi > 1\rangle, \langle a : \bot > x\rangle$, or $\langle a : \top < x\rangle$, or S contains a conjugated pair of fuzzy constraints as in Table 1.

Our calculus for solving the unsatisfiability problem in $\mathcal{ALC}_{FL}$ consists of transformation rule RT1 and a set of constraint propagation rules. All rules except the ones that handle concept modifiers are similar to those for fuzzy $\mathcal{ALC}$, but not as complicated as those for $\mathcal{L}$-$\mathcal{ALC}$. The reason is that in a general lattice, for incomparable elements u and v we have $u \wedge v, u \vee v \notin \{u, v\}$, but in $\mathcal{ALC}_{FL}$ we always have $u \wedge v, u \vee v \in \{u, v\}$ because the HA in consideration is linear. Therefore, rules handling $\neg, \sqcap, \sqcup, \forall$, and $\exists$ constructors are obtained by replacing n by σc, $1-n$ by $\sigma\bar{c}$, where $\bar{c} = -c$, in those rules for fuzzy $\mathcal{ALC}$. Due to the lack of space, we do not show all of them here but some examples. The complete set of rules can be found in [2].

	$\langle \alpha \leq y\rangle$	$\langle \alpha < y\rangle$
$\langle \alpha \geq x\rangle$	$x > y$	$x \geq y$
$\langle \alpha > x\rangle$	$x \geq y$	$x \geq y$

Table 1: Conjugated pairs

$$\langle w : \neg C \geq \sigma c\rangle \rightarrow \langle w : C \leq \sigma\bar{c}\rangle \qquad (\neg_{\geq})$$

$$\langle w : C \sqcap D \geq \sigma c\rangle \rightarrow \langle w : C \geq \sigma c\rangle, \langle w : D \geq \sigma c\rangle \qquad (\sqcap_{\geq})$$

$$\langle w_1 : \exists R.C \leq \sigma c\rangle, \psi \rightarrow \langle w_2 : C \leq \sigma c\rangle \qquad (\exists_{\leq})$$

if ψ is conjugated to $\langle (w_1, w_2) : R \leq \sigma c\rangle$

$$\langle w : \forall R.C \leq \sigma c\rangle \rightarrow \langle (w, x) : R \geq \sigma\bar{c}\rangle, \langle x : C \leq \sigma c\rangle \qquad (\forall_{\leq})$$

if x is a new variable and there is no w' such that both $\langle (w, w') : R \geq \sigma\bar{c}\rangle$ and $\langle w' : C \leq \sigma c\rangle$ are already in the constraint set.

For concept modifiers, we add four new rules $(\delta_{\geq}), (\delta_{>}), (\delta_{\leq})$ and $(\delta_{<})$ as follows:

$$\langle w : \delta C \geq \sigma c\rangle \rightarrow \langle w : C \geq \sigma\delta c\rangle \qquad (\delta_{\geq})$$

$$\langle w : \delta C > \sigma c\rangle \rightarrow \langle w : C > \sigma\delta c\rangle \qquad (\delta_{>})$$

$$\langle w : \delta C \leq \sigma c\rangle \rightarrow \langle w : C \leq \sigma\delta c\rangle \qquad (\delta_{\leq})$$

$$\langle w : \delta C < \sigma c\rangle \rightarrow \langle w : C < \sigma\delta c\rangle \qquad (\delta_{<})$$

Proposition 12. *A finite set S of fuzzy constraints is satisfiable iff there exists a clash-free completion of S.*

Proof. Because the HA representing the truth domain is linear, the argument is similar to the ones for fuzzy $\mathcal{ALC}$ in [8] except for the rules handling hedge modifiers. Hence, we focus on these rules only in order to save space.

$(\Rightarrow)$ By case analysis, it is easily verified that the rules are sound, i.e., if we apply a rule to a satisfiable set S_1 of constrants, the result S_2

is also satisfiable, and thus, clash-free. Let us consider the rule $(\delta_{\geq})$, for rules $(\delta_{>}), (\delta_{<})$ and $(\delta_{\leq})$, similar arguments can be used.

$(\delta_{\geq})$ Assume that $(\delta_{\geq})$ is applicable, i.e., S_1 contains $\langle w : \delta C \geq \sigma c\rangle$ for some $w, \sigma c$, and modified concept δC. Since S_1 is satisfiable, there exist an interpretation $\mathcal{I}$ that satisfies $\langle w : \delta C \geq \sigma c\rangle$, i.e., $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) \geq \sigma c$. Let $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0 \geq \sigma c$, $\delta = h_p \dots h_1$ where $p \geq 0, h_i \in H \forall i = 1 \dots p$. Applying RT1 p times, we have $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma h_p \dots h_1 c_0 = \gamma \delta c_0$. There are two cases: $\gamma c_0 = \sigma c$ or $\gamma c_0 > \sigma c$. In the first case, $\gamma c_0 = \sigma c$ implies $\gamma \delta c_0 = \sigma \delta c$. In the second case, according to Corollary 8, $\gamma c_0 > \sigma c \Leftrightarrow \gamma \delta c_0 > \sigma \delta c$. Therefore, we always have $\gamma \delta c_0 \geq \sigma \delta c$. Hence, $C^{\mathcal{I}}(w^{\mathcal{I}}) \geq \sigma \delta c$, i.e., $\mathcal{I}$ satisfies $\langle w : C \geq \sigma \delta c\rangle$ and S_2 as well.

$(\Leftarrow)$ Assume S' is a clash-free completion of S. Let us construct a model for the fuzzy constraints in S' that contains only primitive concepts or roles, and prove that it is a model of S', and S as well.

Since S' is clash-free, for each concept A and element w that appear in S' in the form $\langle w : A \circ x\rangle$, there exists a non-empty set $\tau(A, w) = \{x_0 \in X \mid \forall \langle w : A \circ x\rangle \in S'.x_0 \circ x\}$ in which $\circ \in \{<, \leq, \geq, >\}$. Similarly, for each role R and pair (w_1, w_2) that appear in S' in the form $\langle (w_1, w_2) : R \circ x\rangle$, there exists a non-empty set $\tau(R, w_1, w_2) = \{x_0 \in X \mid \forall \langle (w_1, w_2) : R \circ x\rangle \in S'.x_0 \circ x\}$ in which $\circ \in \{<, \leq, \geq, >\}$.

Consider an interpretation $\mathcal{I}$ such that the domain $\Delta^{\mathcal{I}}$ is the set of objects appearing in S', $\forall w \in \Delta^{\mathcal{I}}.w^{\mathcal{I}} = w$, and $A^{\mathcal{I}}(w^{\mathcal{I}}) \in \tau(A, w), R^{\mathcal{I}}({w_1}^{\mathcal{I}}, {w_2}^{\mathcal{I}}) \in \tau(R, w_1, w_2)$.

It is easily verified that this interpretation satifies all constraints for primitive concepts and roles in S' if S' is clash-free. The satisfaction of the other fuzzy constraints in S' are shown by induction on the structure of the $\mathcal{ALC}_{FL}$ -formula in the constraints. Once again, let us just represent one case for space reasons.

Case $\langle w : \delta C > \sigma c\rangle$ Because S' is complete, $\langle w : C > \sigma \delta c\rangle$ is in S' and is satisfied by $\mathcal{I}$ by induction assumption, i.e., $C^{\mathcal{I}}(w^{\mathcal{I}}) > \sigma \delta c$. Let $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0 > \sigma \delta c$. According to Proposition 10, we have $\gamma c_0 > \sigma \delta c \Rightarrow \delta^{-}(\gamma c_0) > \delta^{-}(\sigma \delta c)$. Since $C^{\mathcal{I}}(w^{\mathcal{I}}) = \gamma c_0$, we have$(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \delta^{-}(\gamma c_0)$. Besides, $\delta^{-}(\sigma \delta c) = \sigma c$. Hence, $(\delta C)^{\mathcal{I}}(w^{\mathcal{I}}) = \delta^{-}(\gamma c_0) > \delta^{-}(\sigma \delta c) = \sigma c$. Therefore, $\mathcal{I}$ satisfies $\langle w : \delta C > \sigma c\rangle$. □

Let us close the section by an example to demonstrate how the calculus works.

Example 13. *Consider a knowledge base* Σ*:*

"A car is a sport car if it is very likely that it can run very very fast" holds to a degree at least True. In particular, for Audi_TT cars:

$$\langle tt : \neg(\exists speed.VVFast) \sqcup Sport \geq True\rangle \quad (5)$$

"An Audi_TT car can run at 250km/h" holds to a degree more than more-or-less True:

$$\langle (tt, 250) : speed \geq MolTrue\rangle \quad (6)$$

"250km/h is fast" holds to a degree at least More True:

$$\langle 250 : Fast \geq MTrue\rangle \quad (7)$$

We want to prove that Σ *entails that "Audi_TT cars are sport cars" to a degree more than Probably True. That is,* Σ *together with (8) is unsatisfiable.*

$$\langle tt : Sport < PTrue\rangle \quad (8)$$

The rule $(\sqcup_{\geq})$ *gives two choices:*

$$\langle tt : \neg(\exists speed.VVFast) \geq True\rangle \quad (9)$$

$$\langle tt : Sport \geq True\rangle \quad (10)$$

The latter immediately yields a clash with (8). The application of the rule $(\neg_{\geq})$ *on the former one gives:*

$$\langle tt : (\exists speed.VVFast) \leq False\rangle \quad (11)$$

Since (6) is conjucated to $\langle (tt, 250) : speed \leq False\rangle$*, rule* $(\exists_{\leq})$ *yields:*

$$\langle 250 : VVFast \leq False\rangle \quad (12)$$

Rule $(\delta_{\leq})$ *applying on (12) yields*

$$\langle 250 : Fast \leq VVFalse\rangle \quad (13)$$

which clashes with (7).

Hence, there is no clash-free completion of $\Sigma \cup \{(8)\}$*, i.e.,* $\Sigma \cup \{(8)\}$ *is unsatisfiable. Therefore,* $\Sigma \models \langle tt : Sport \geq PTrue\rangle$*.*

5 Conclusions

In the paper, we have presented the fuzzy linguistic DL $\mathcal{ALC}_{FL}$, where the truth domain is represented by a monotonic HA. The main feature is that an element a belongs to a concept C with a degree specified by a linguistic value, which itself is part of a lattice represented by a HA. Besides, $\mathcal{ALC}_{FL}$ allows the modification by hedges. Furthermore, the ambiguity of the hedge application in $\mathcal{ALC}_{FH}$ and $\mathcal{ALC}_{FLH}$ is solved in $\mathcal{ALC}_{FL}$. To the best of our knowledge, no DL that not only allows the modification by hedges but also computes directly with words has been proposed.

A sound and complete decision procedure for the satisfiability problem in $\mathcal{ALC}_{FL}$ has been also presented. In the future, we plan to consider the subsumption problem in $\mathcal{ALC}_{FL}$.

Note that in this paper, we restrict ourselves to linear HAs in order to define inverse mappings of hedges with ease. In the general case of HAs [7, 5], where there are some incomparable hedges in H^+ or H^-, e.g., *Possibly* with *Approximately*, the domain becomes a lattice instead of linear. Thus, our further work is to extend our logic to the case where the truth domain is not linear.

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